LA-UR-21-24962
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Title: Quantum Linear Systems Problem

Author(s): Subasi, Yigit

Intended for: I am invited to speak at LANL/Texas A&M University Quantum Science Seminar Series.

Issued: 2021-05-26 (rev.1)

Los Alamos NATIONAL LABORATORY

Quantum linear systems problem

Yigit Subasi

Computer, Computational, and Statistical Sciences Division

LANL/Texas A&M University
Quantum Science Seminar Series

05/25/2021

LA-UR-21-24962

LDRD innovation for our nation

U.S. DEPARTMENT OF ENERGY
Office of Science

NNSA National Nuclear Security Administration

Talk Overview

01	02	03	04
Introduction & Motivation	**An AQC-inspired algorithm**	**A variational algorithm**	**Complexity of verification**

Introduction & Motivation

$$\begin{bmatrix} A_{11} & A_{12} & \cdots \\ \vdots & \ddots & \\ A_{N1} & & A_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$$

$$|x\rangle = \frac{\sum_{i=1}^{N} x_i \, |i\rangle}{\sqrt{\sum_{i=1}^{N} |x_i|^2}}$$

System of linear equations

Given an $N \times N$ matrix A, an $N-$ dimensional vector $\vec{b}$ and the equation

$$\begin{bmatrix} A_{11} & A_{12} & \dots \\ \vdots & \ddots & \\ A_{N1} & & A_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$$

that is $A\vec{x} = \vec{b}$, solve for $\vec{x} = A^{-1}\vec{b}$.

The best general purpose classical algorithm (Conjugate Gradient) has complexity $O(N\kappa)$, where $\kappa = \|A\|\|A^{-1}\|$ is the condition number of A.

Quantum linear systems problem (QLSP)

Given the classical system of linear equations prepare an ϵ-approximation of a quantum state

Harrow, Hassidim, Lloyd
PRL 103, 150502 (2009)

$$|x\rangle = \frac{\sum_{i=1}^{N} x_i |i\rangle}{\sqrt{\sum_{i=1}^{N} |x_i|^2}}$$

$\log(N)$ qubits sufficient.

where $A\vec{x} = \vec{b}$, $\vec{x} = (x_1, x_2, \ldots, x_N)$.

Good for obtaining global properties of the solution via $x^T \mathcal{O} x$.

Best algorithms for this problem have complexity: $\tilde{O}\left(\log(N)\kappa\right)$.

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where $A\vec{x} = \vec{b}$, $\vec{x} = (x_1, x_2, \ldots, x_N)$.

The approximate (pure or mixed) quantum state ρ_x satisfies

$$\frac{1}{2} \mathrm{Tr} \, |\rho_x - |x\rangle\langle x|| \leq \epsilon$$

Quantum linear systems problem (QLSP)

Assumptions:

- A is invertible: condition number $\kappa < \infty$
- $\|A\| \leq 1$.
- A is Hermitian (trivial generalization to non-Hermitian).
- Sparse A_{ij} can be computed efficiently (or e^{iAt} implemented efficiently)

Let U_b be a procedure that prepared the state

$$U_b |0\rangle = |b\rangle \;\; ; \;\; |b\rangle = \frac{\sum_{i=1}^{N} b_i |i\rangle}{\sqrt{\sum_{i=1}^{N} |b_i|^2}}$$

Will be treated as black-boxes.

Potential & Caveats

BQP-Complete -> Potential for exponential quantum speed-up!

Relevant to many fields: Solving differential equations, quantum machine learning, large sparse network problems,...

BUT!

A number of assumptions must be made in order to obtain quantum speedups. These assumptions include:

- efficient preparation of certain states, i.e. $U_b \left|0\right\rangle = \left|b\right\rangle$
- nice scaling of the condition number
- solving certain problems like computing expectation values

Aaronson (2015) "Read the fine print"

For these reasons, shown quantum speedups are typically polynomial.

Applications

- Determining the quality of a least-squares fit. Wiebe *et al*. (2012)
- Analyzing large sparse electrical networks. Wang (2013)
- Solving systems of linear ordinary differential equation. Berry *et al*. (2017)
- Estimating the hitting time of a Markov chain. Chowdhury *et al*. (2017)
- Computing electromagnetic scattering cross section of a target. Clader *et al*. (2013)
  - Resource analysis of HHL algorithm by Scherer *et al*. (2017):

$N = 332,020,680,\ \epsilon = 0.01,\ 341$ qubits, 10^{25} gates, and 10^{21} measurements.

QLSP Algorithms

Previous Results

- Harrow, Hassidim, Lloyd (2008)
$$\tilde{O}(\kappa^2 \log(N)/\epsilon)$$
- Ambainis (2012)
$$\tilde{O}(\kappa \log(N)/\epsilon^3)$$
- Childs, Kothari, Somma (2017)
$$\tilde{O}(\kappa \log(N) \mathrm{poly} \log(1/\epsilon))$$

Main Subroutines

- Hamiltonian simulation
- Phase estimation / Linear Combination of Unitaries
- Amplitude Amplification / Variable-time Amplitude Amplification

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Main Subroutines

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Within two weeks of posting our result, Wen et al. PRA (2019) implemented our algorithm in NMR to solve an 8x8 QLSP.

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- An, Lin (2019)

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Main Subroutines

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Variational Algorithms

- Bravo-Prieto et al. (19)
- Yuan et al. (19)
- Rebentrost et al. (19)
- An, Lin (19)

Geared towards NISQ devices.
Heuristic algorithms, scaling poorly understood.

An AQC-inspired algorithm

PHYSICAL REVIEW LETTERS **122,** 060504 (2019)

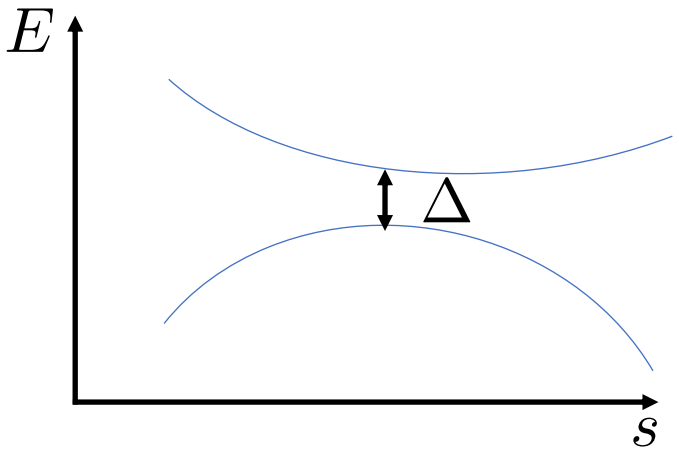
Quantum Algorithms for Systems of Linear Equations Inspired by Adiabatic Quantum Computing

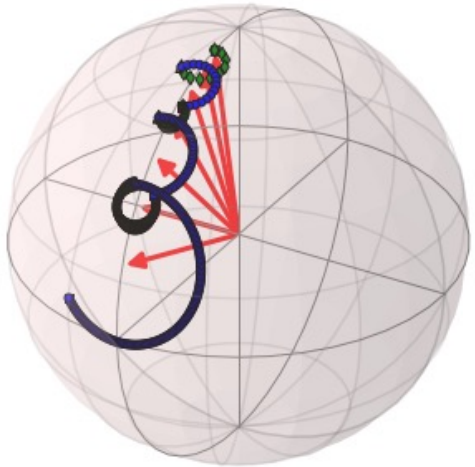
Yiğit Subaşı and Rolando D. Somma
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

Davide Orsucci
Department of Theoretical Physics, University of Innsbruck, Innsbruck 6020, Austria

(Received 7 June 2018; published 14 February 2019)

We present two quantum algorithms based on evolution randomization, a simple variant of adiabatic quantum computing, to prepare a quantum state $|x\rangle$ that is proportional to the solution of the system of linear equations $A\vec{x} = \vec{b}$. The time complexities of our algorithms are $O(\kappa^2 \log(\kappa)/\epsilon)$ and $O(\kappa \log(\kappa)/\epsilon)$, where κ is the condition number of A and ϵ is the precision. Both algorithms are constructed using families of Hamiltonians that are linear combinations of products of A, the projector onto the initial state $|b\rangle$, and single-qubit Pauli operators. The algorithms are conceptually simple and easy to implement. They are not obtained from equivalences between the gate model and adiabatic quantum computing. They do not use phase estimation or variable-time amplitude amplification, and do not require large ancillary systems. We discuss a gate-based implementation via Hamiltonian simulation and prove that our second algorithm is almost optimal in terms of κ. Like previous methods, our techniques yield an exponential quantum speed-up under some assumptions. Our results emphasize the role of Hamiltonian-based models of quantum computing for the discovery of important algorithms.





Adiabatic Quantum Computing

(Our starting point is AQC, but we will use a variant in our algorithm.)

At initial time prepare the ground state of a simple Hamiltonian

$$H_i \left|\psi_i\right\rangle = E_i \left|\psi_i\right\rangle$$

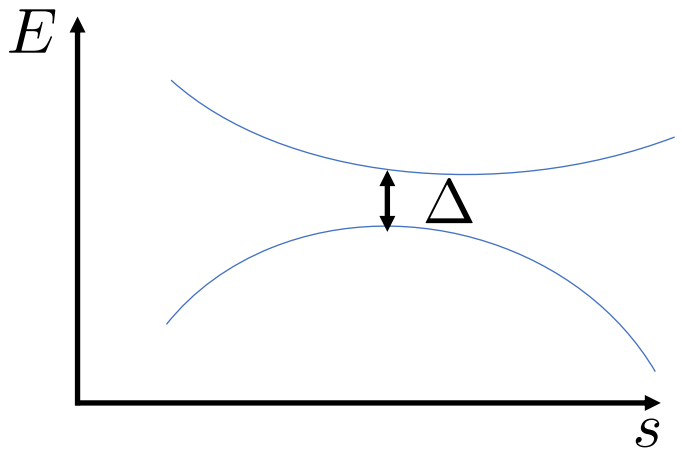
Design a final Hamiltonian whose ground state is the desired output of the computation.

$$H_f \left|\psi_f\right\rangle = E_f \left|\psi_f\right\rangle$$

Interpolate from H_i to H_f with $H(s)$ s.t. $H(0) = H_i$ & $H(1) = H_f$.

Adiabatic theorem: if the interpolation is done *slow enough* the final ground state $\left|\psi_f\right\rangle$ can be prepared with high probability.

Slow enough means $T \sim O(1/\Delta^2)$



QLSP – Finding the final Hamiltonian

For now assume $A > 0$.

Let $P_b^{\perp} \equiv I - |b\rangle\langle b|$. Observe that

$$A|x\rangle \propto |b\rangle$$
$$\underbrace{P_b^{\perp} A}_{B}|x\rangle = 0$$

Define $H_f = B^{\dagger}B = AP_b^{\perp}A$.

- Hermitian
- Positive-semidefinite $\implies |x\rangle$ is a GS
- GS is unique

QLSP – Finding an interpolation

Define $A(s)$ which interpolates between identity and A

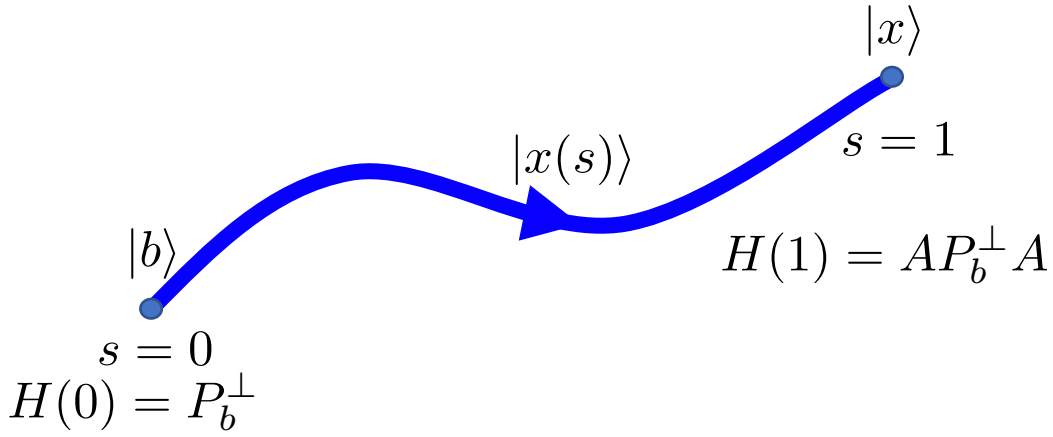
$$A \longrightarrow A(s) \equiv (1-s)I + sA$$

The parametrized Hamiltonian becomes $H(s) \equiv A(s)P_b^{\perp}A(s)$

And the eigenpath is $|x(s)\rangle \propto A(s)^{-1}|b\rangle$

That is, for each value of s , the GS encodes the solution of the LSP

$$A(s)\vec{x}(s) = \vec{b}$$



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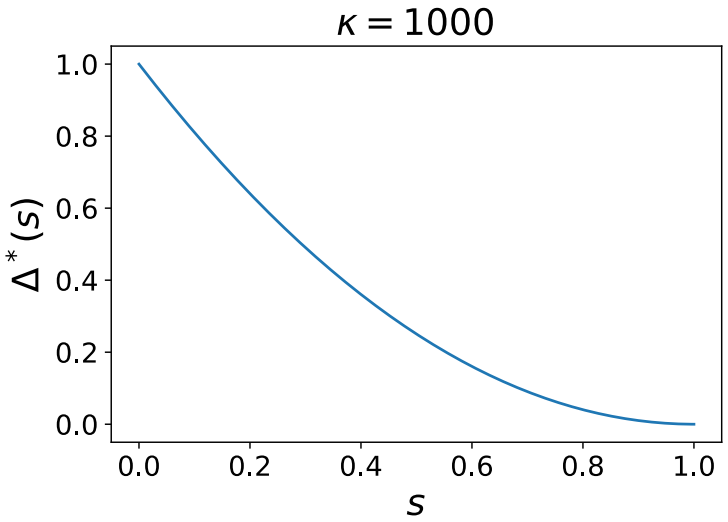
GAP ANALYSIS:

$$1/\kappa^2 \leq \underbrace{(1-s+s/\kappa)^2}_{\Delta^*(s)} \leq \Delta(s)$$

$\Delta^*(s)$: lower bound on the gap



Spectral gap amplification (Somma, Boixo 2013)

Time complexity of algorithms based on adiabatic evolution is dictated by the minimum gap.

By increasing the gap we can reduce the complexity.

$$B(s) = P_b^{\perp} A(s)$$

$$H(s) = B^{\dagger}(s) B(s)$$

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$$H'(s) = \underbrace{\sigma^{-}}_{\text{ancilla register}} \otimes B(s) + \sigma^{+} \otimes B(s)^{\dagger} = \begin{pmatrix} 0 & B(s)^{\dagger} \\ B(s) & 0 \end{pmatrix}$$

$$(H'(s))^2 = |0\rangle\langle 0| \otimes H(s) + |1\rangle\langle 1| \otimes B(s)B(s)^{\dagger} = \begin{pmatrix} H(s) & 0 \\ 0 & B(s)B(s)^{\dagger} \end{pmatrix}$$

$$\boxed{\langle 0| \left(H'(s)\right)^2 |0\rangle = H(s)}$$

The gap of H' is quadratically bigger than that of H.

Spectral gap amplification (Somma, Boixo 2013)

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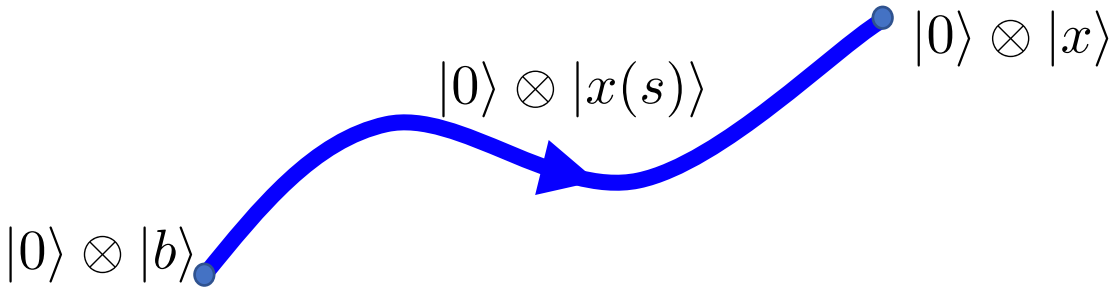
$$H'(s) = \begin{pmatrix} 0 & A(s)P_b^{\perp} \\ P_b^{\perp}A(s) & 0 \end{pmatrix}$$



Eigenpath traversal

One can use any method to follow the eigenstates of Hamiltonians $H(s)$ and $H'(s)$.

Using adiabatic evolution, the time complexity can be upper-bounded by $O(1/\Delta^2)$

$H(s)$	$H'(s)$
$\frac{1}{\kappa^2} \leq \Delta(s)$	$\frac{1}{\kappa} \leq \Delta'(s)$
$T = O(\kappa^4)$	$T = O(\kappa^2)$

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We achieve $T = \tilde{O}(\kappa^2)$ $T = \tilde{O}(\kappa)$ using the randomization method.

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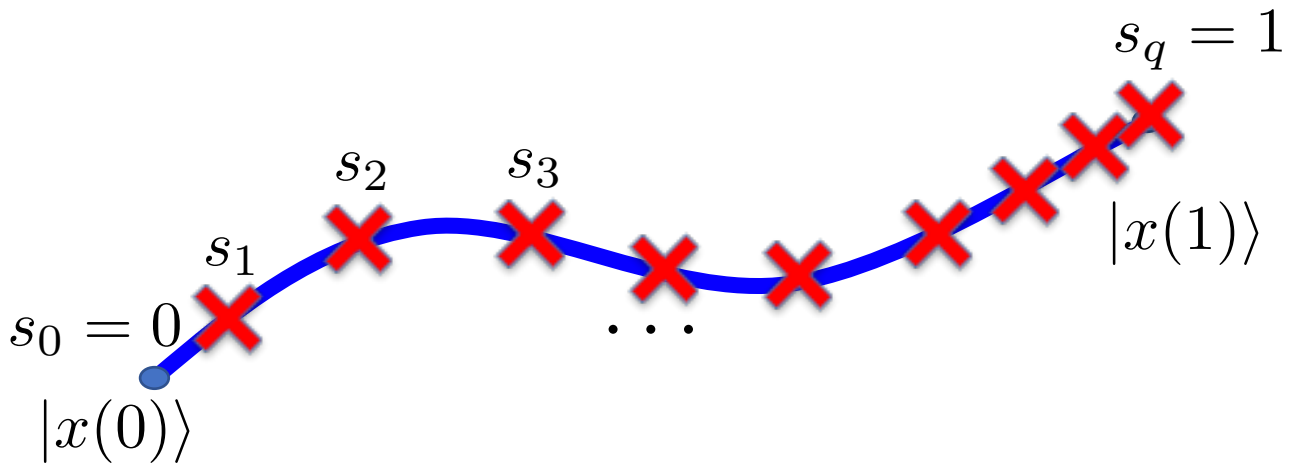
We achieve $T = \tilde{O}(\kappa^2)$ $T = \tilde{O}(\kappa)$ using the randomization method.

(A careful analysis by An, Lin (2019) obtained the same scaling with AQC by choosing a proper scheduling function.)

Randomization Method

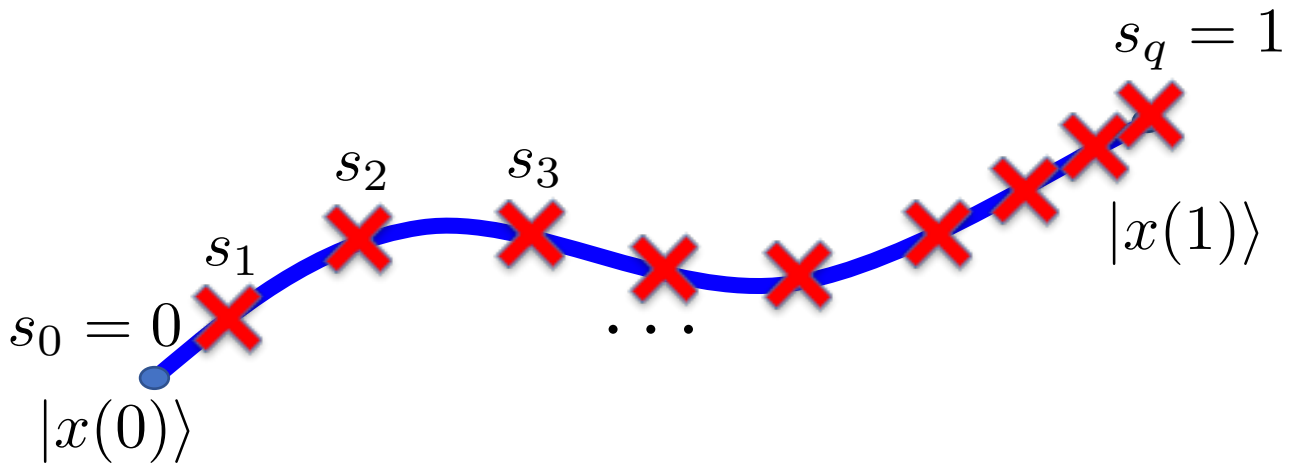
(Boixo, Knill, Somma 2009)



Quantum Zeno Effect: By performing a sequence of energy measurements wrt sufficiently close Hamiltonians, we can stay in the corresponding eigenspace.

Randomization Method

(Boixo, Knill, Somma 2009)



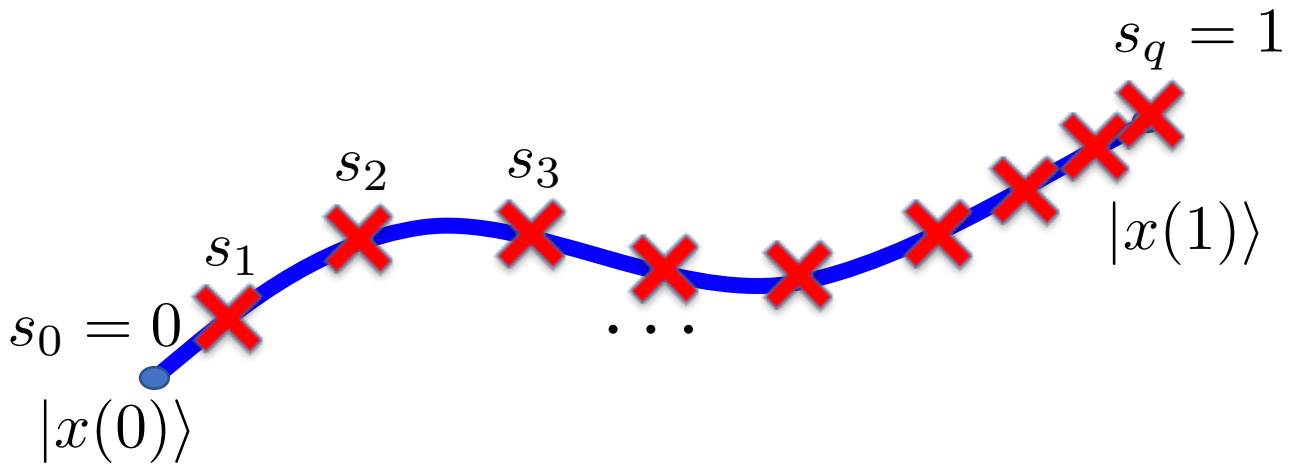
Quantum Zeno Effect: By performing a sequence of energy measurements wrt sufficiently close Hamiltonians, we can stay in the corresponding eigenspace.

Imperfect measurements can be simulated by evolving with the corresponding Hamiltonian for random time. This reduces coherences between eigenstates and thereby simulates a measurement.

$$\rho = \sum_{n,n'} \rho_{nn'} \, |E_n\rangle\langle E_{n'}| \longrightarrow \sum_{n,n'} \rho_{nn'} \left(\int \mathrm{d}t \, \Pr(t) e^{-i(E_n - E_{n'})t} \right) |E_n\rangle\langle E_{n'}|$$

Randomization Method

(Boixo, Knill, Somma 2009)



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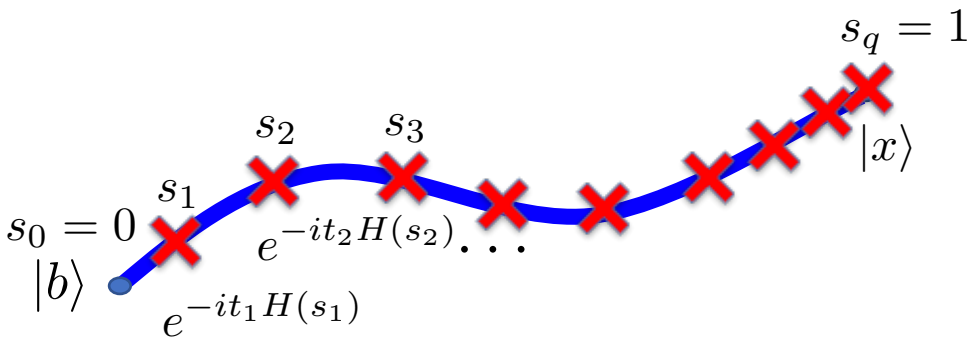
$$\mathrm{Unif}\,[0, 2\pi/\Delta]$$

Algorithm 1



Given condition number κ and precision ϵ

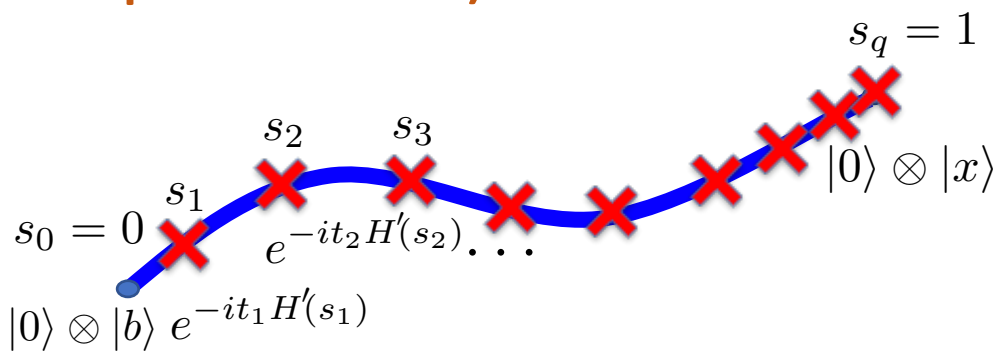
- Set $q = \Theta(\log^2(\kappa)/\epsilon)$
- For $j = 1, \ldots, q$, let $s_j = \dfrac{1 - \kappa^{-j/q}}{1 - \kappa^{-1}}$

  and t_j be sampled from the uniform distribution $\left[0, \dfrac{2\pi}{\Delta^*(s_j)}\right]$

  Bound on the gap
- Apply $e^{-it_q H(s_q)} \ldots e^{-it_1 H(s_1)}$ to $|b\rangle$.

The average time complexity is $T := \sum_{j=1}^{q} \langle t^j \rangle = O\left(\kappa^2 \log(\kappa)/\epsilon\right).$

Algorithm 2 (with spectral gap amplification)



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- Set $q = \Theta(\log^2(\kappa)/\epsilon)$
- For $j = 1, \ldots, q$, let $s_j = \dfrac{1 - \kappa^{-j/q}}{1 - \kappa^{-1}}$

  and t_j be sampled from the uniform distribution $\left[0, \dfrac{2\pi}{\sqrt{\Delta^*(s_j)}}\right]$

  Bound on the amplified gap
- Apply $e^{-it_q H'(s_q)} \ldots e^{-it_1 H'(s_1)}$ to $|0\rangle \otimes |b\rangle$.

The average time complexity is $T := \sum_{j=1}^{q} \langle t^j \rangle = O\left(\kappa \log(\kappa)/\epsilon\right)$

Example

Let $A|\lambda\rangle = \lambda|\lambda\rangle$,

$|b\rangle \propto |\lambda = 1\rangle + \frac{1}{\kappa}|\lambda = 1/\kappa\rangle$

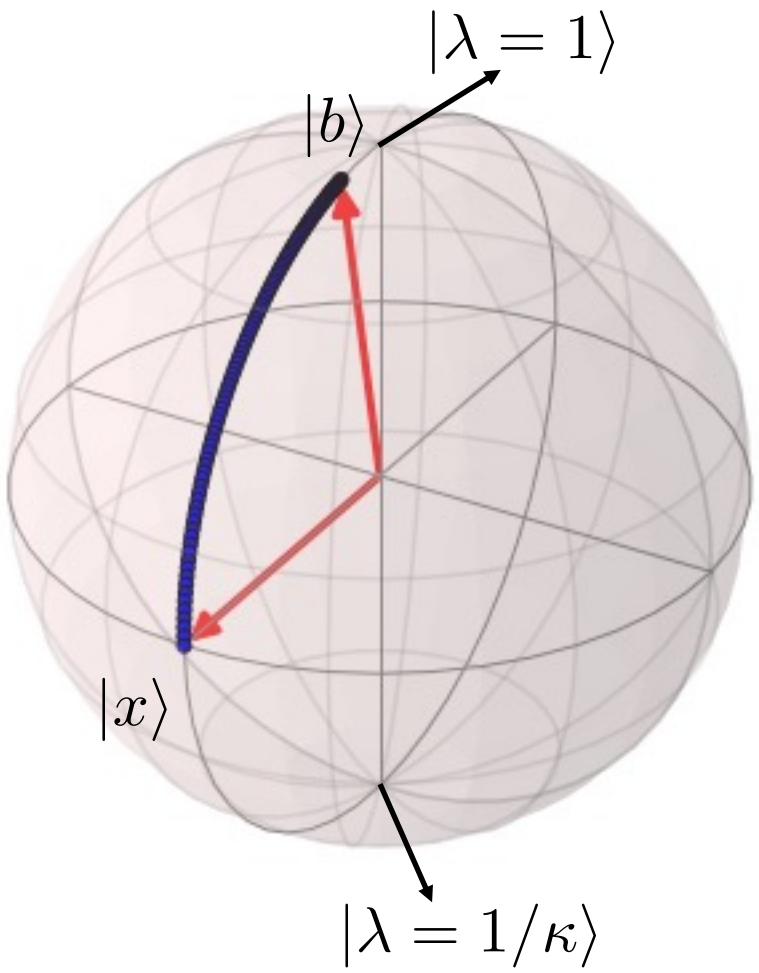
and $\kappa = 10$.

The dynamics is restricted to a two-dimensional Hilbert space.

$|x\rangle \propto |\lambda = 1\rangle + |\lambda = 1/\kappa\rangle$

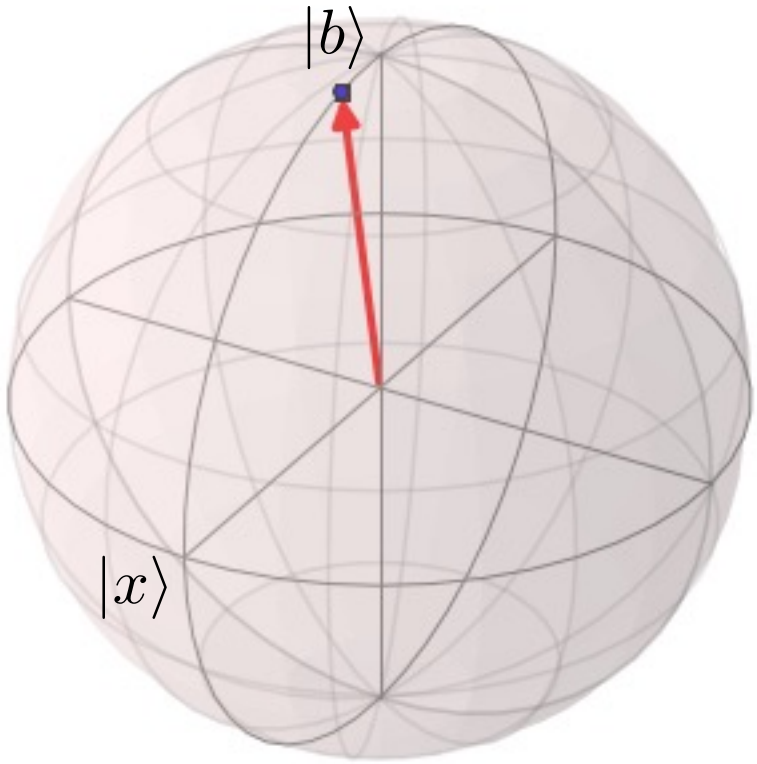
Consider 7 steps of RM:

$$q = 7$$



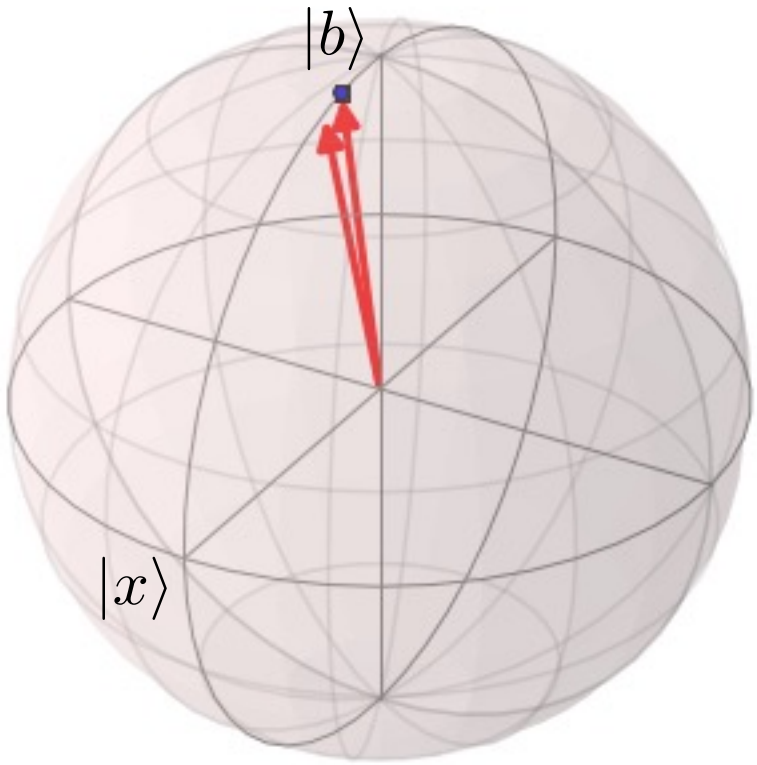
Example

$$s_0 = 0$$



Example

$$s_1 \approx 0.36$$



$$t_1 \in [0, 13.6] \longrightarrow t_1 = 9.74$$

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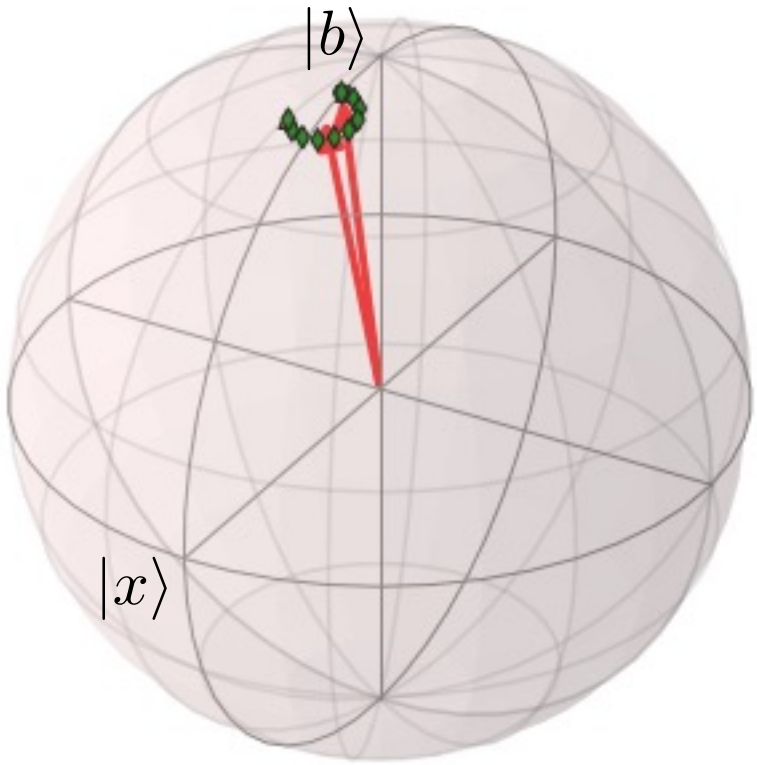

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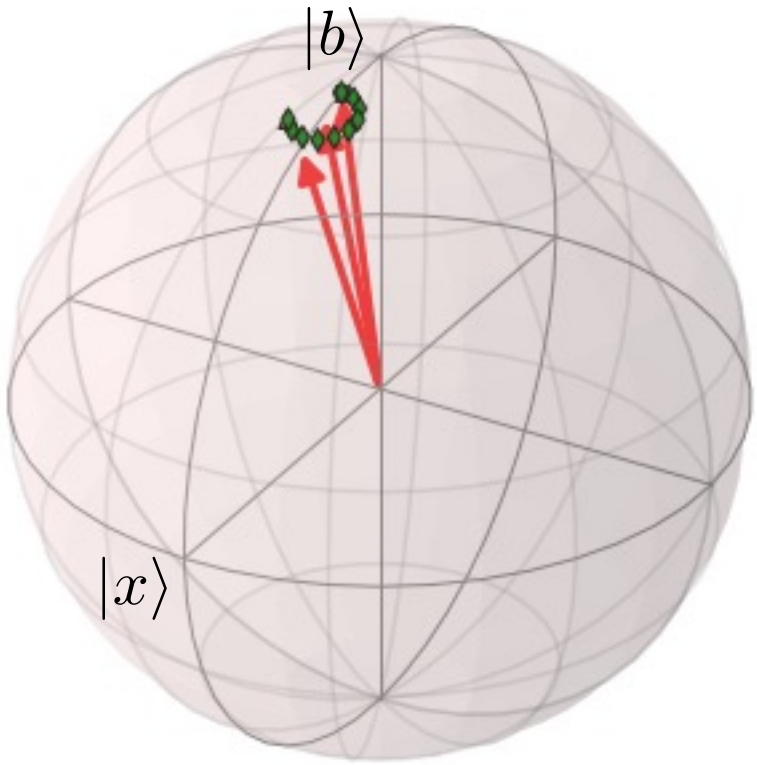
Example

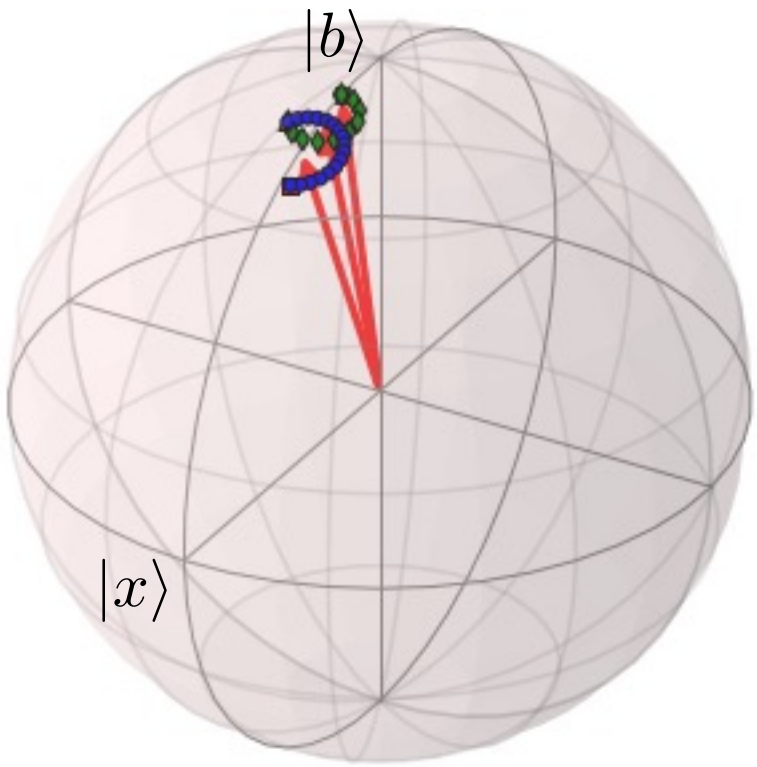
$$s_2 \approx 0.59$$



$$t_2 \in [0, 28.2] \longrightarrow t_2 = 17.0$$

Example

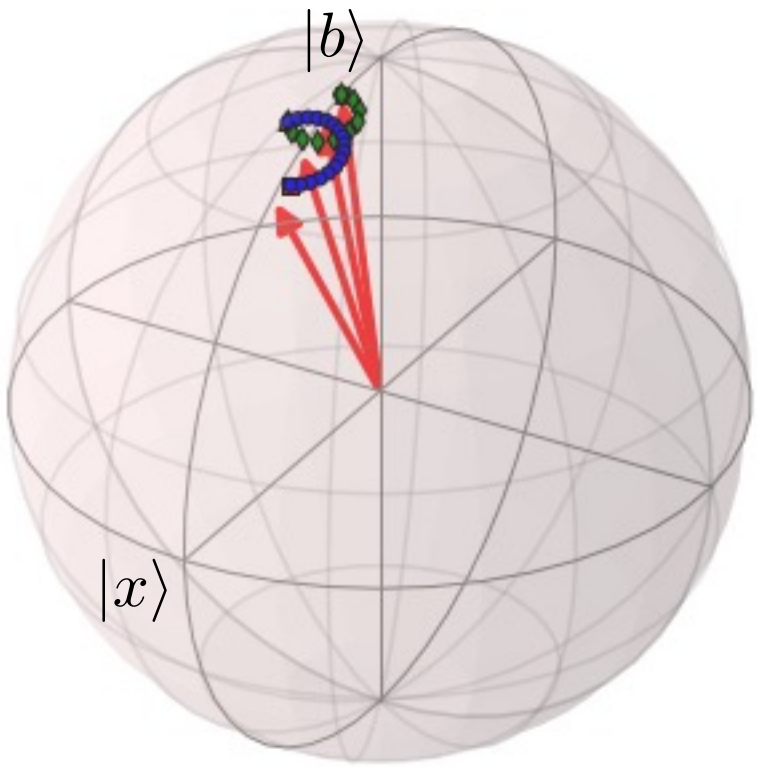
$$s_2 \approx 0.59$$



$$t_2 \in [0, 28.2] \longrightarrow t_2 = 17.0$$

Example

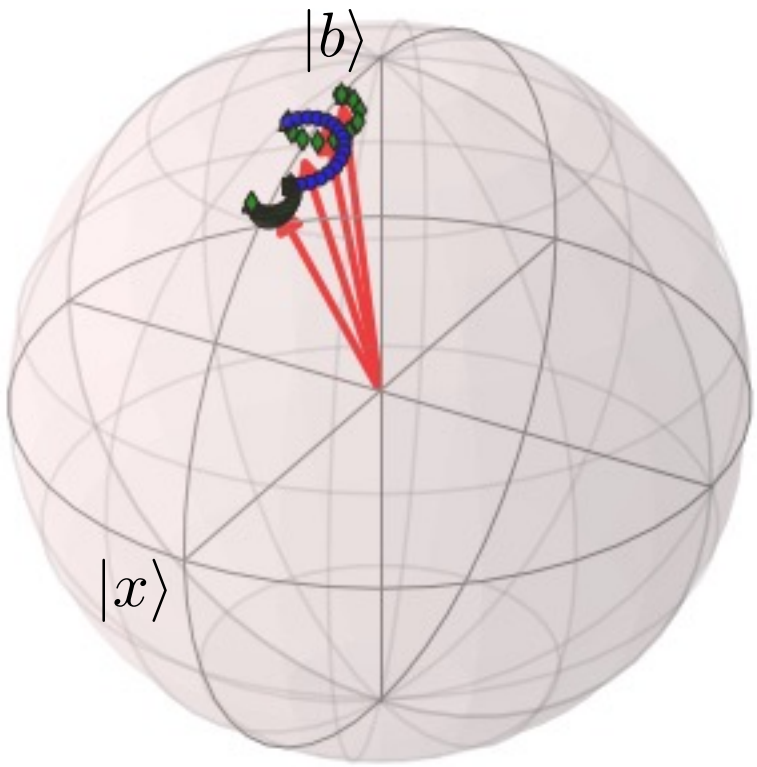
$$s_3 \approx 0.74$$



$$t_3 \in [0, 55.2] \longrightarrow t_3 = 30.1$$

Example

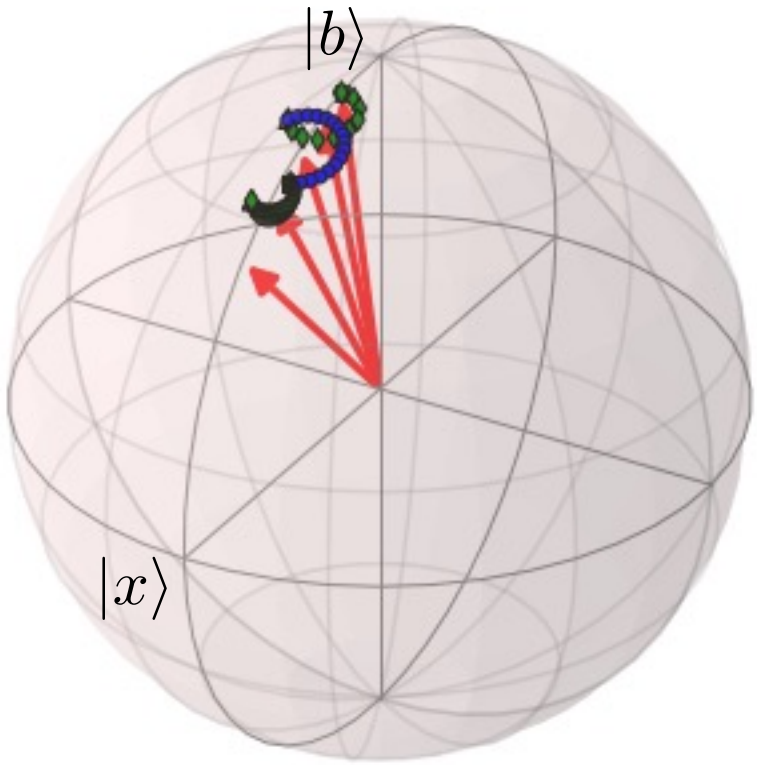
$$s_3 \approx 0.74$$



$$t_3 \in [0, 55.2] \longrightarrow t_3 = 30.1$$

Example

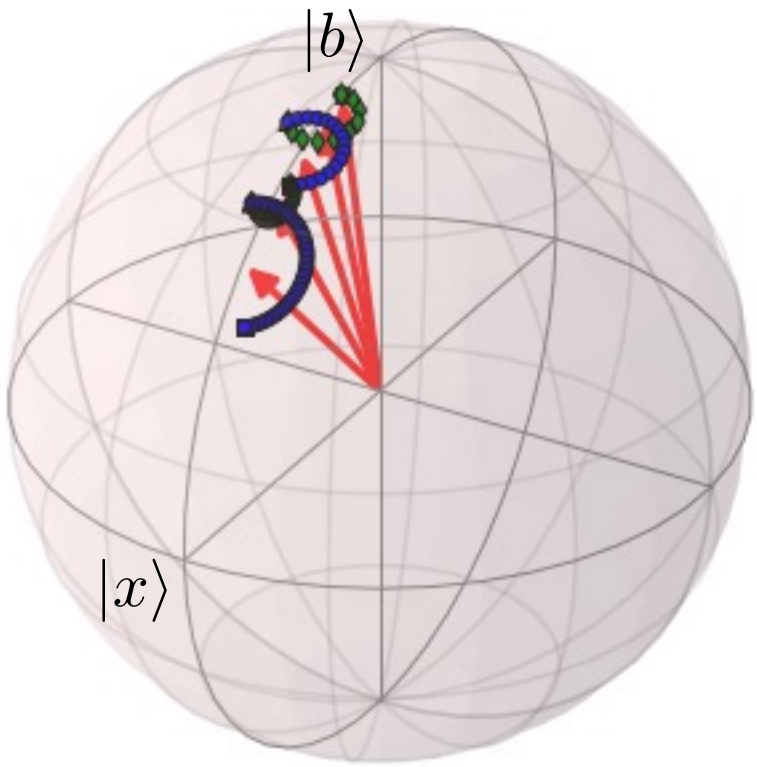
$$s_4 \approx 0.83$$



$$t_4 \in [0, 102.2] \longrightarrow t_4 = 43.3$$

Example

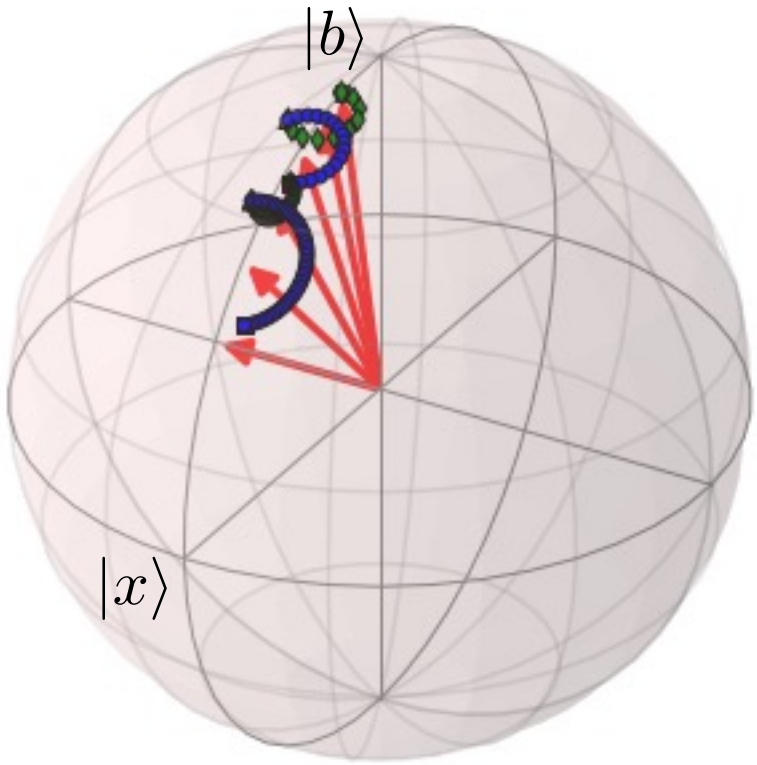
$$s_4 \approx 0.83$$



$$t_4 \in [0, 102.2] \longrightarrow t_4 = 43.3$$

Example

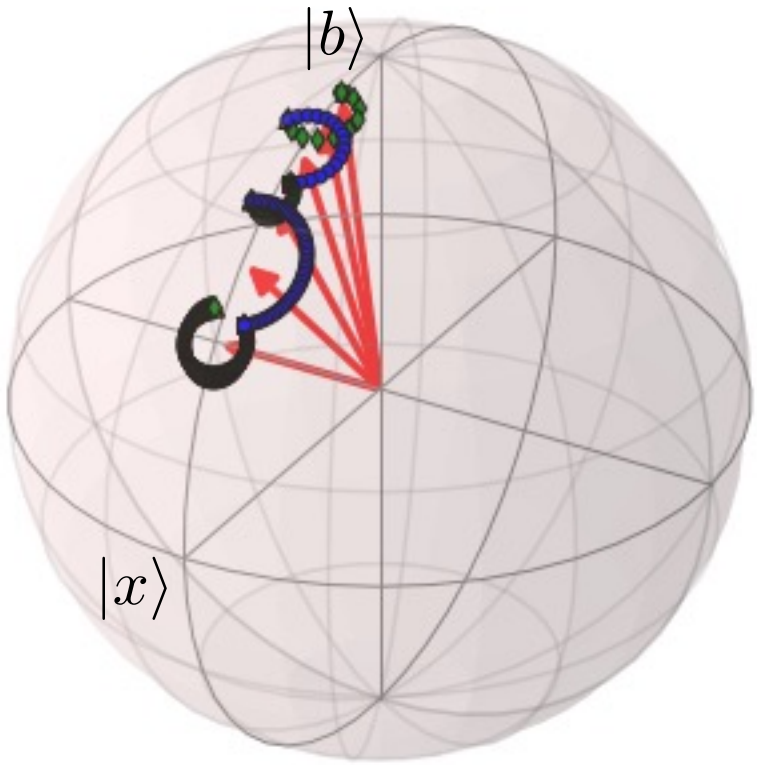
$$s_5 \approx 0.90$$



$$t_5 \in [0, 181.0] \longrightarrow t_5 = 116.9$$

Example

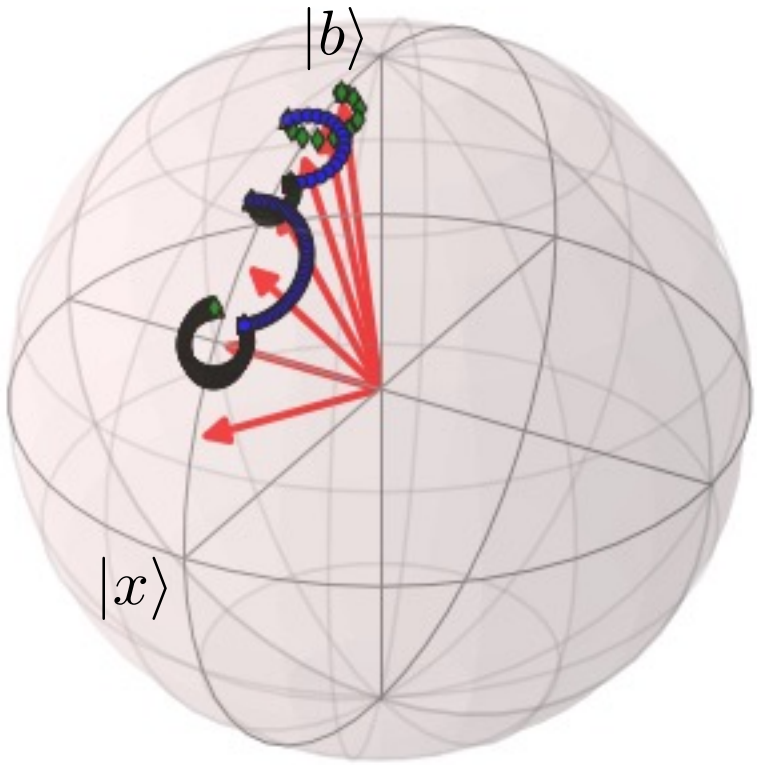
$$s_5 \approx 0.90$$



$$t_5 \in [0, 181.0] \longrightarrow t_5 = 116.9$$

Example

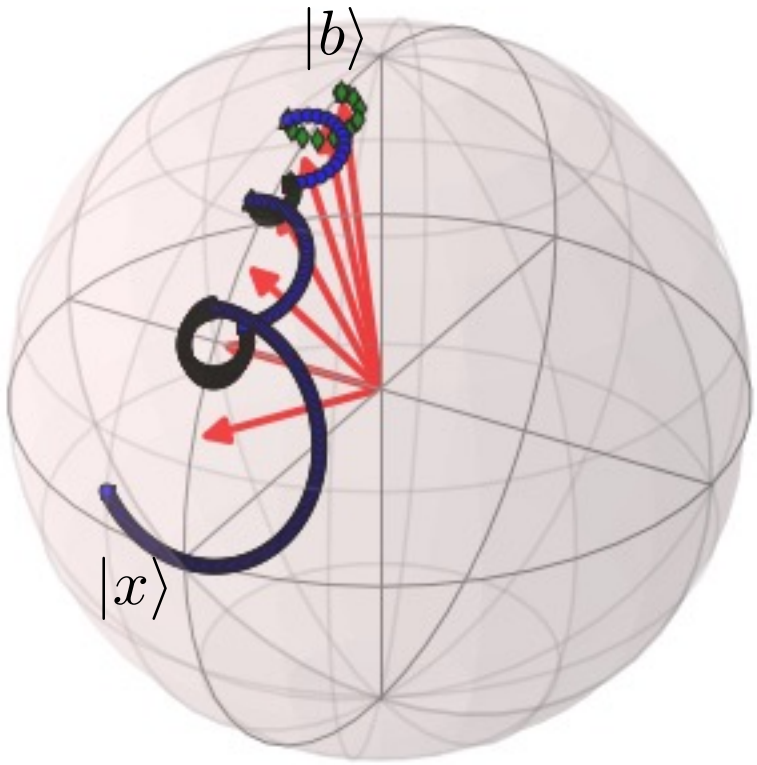
$$s_6 \approx 0.96$$



$$t_6 \in [0, 320.3] \longrightarrow t_6 = 140.2$$

Example

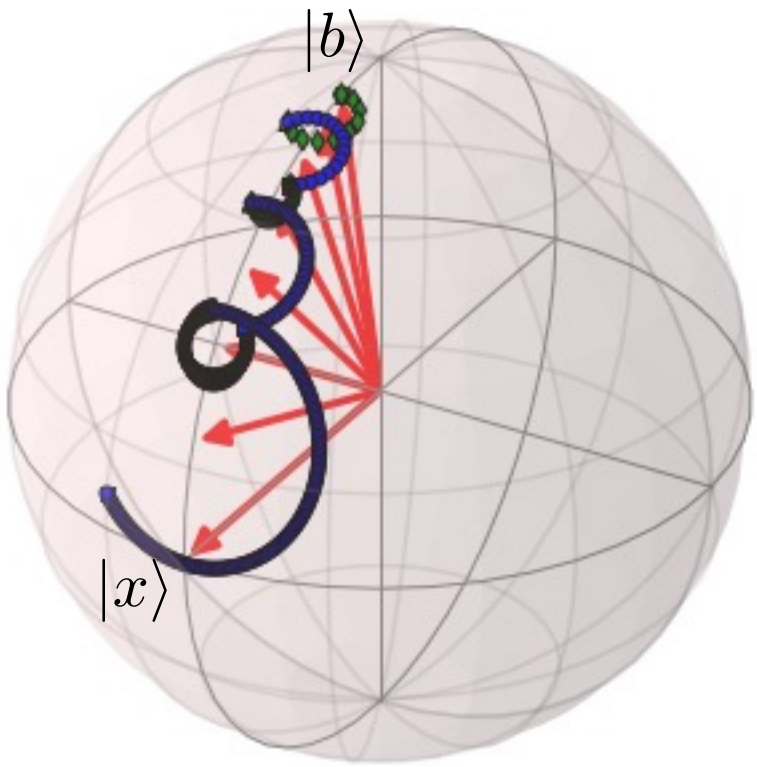
$$s_6 \approx 0.96$$



$$t_6 \in [0, 320.3] \longrightarrow t_6 = 140.2$$

Example

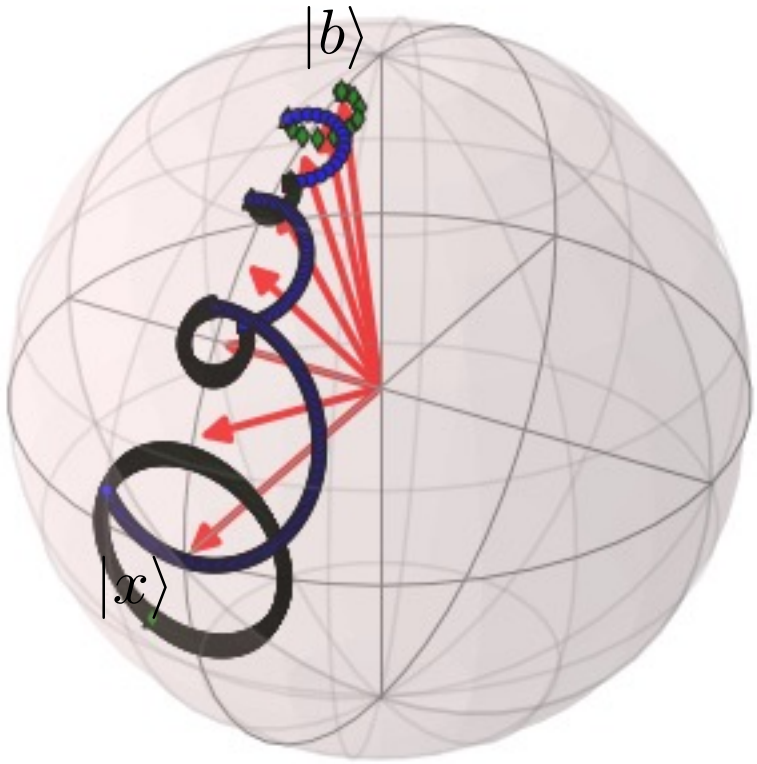
$$s_7 \approx 1.0$$



$$t_7 \in [0, 628.3] \longrightarrow t_7 = 560.3$$

Example

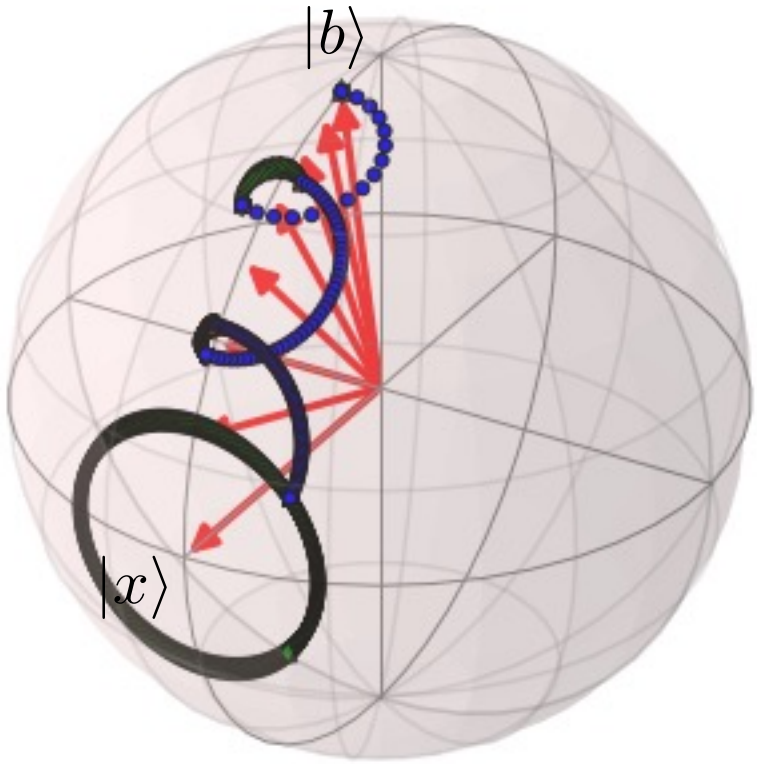
$$s_7 \approx 1.0$$



$$t_7 \in [0, 628.3] \longrightarrow t_7 = 560.3$$

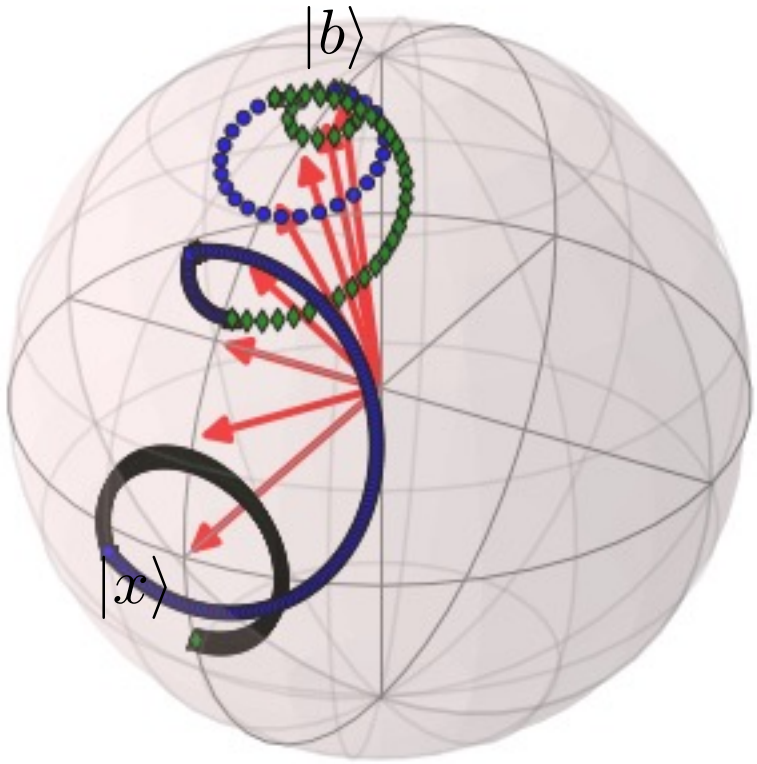
Example

Sample run of the Randomization Method



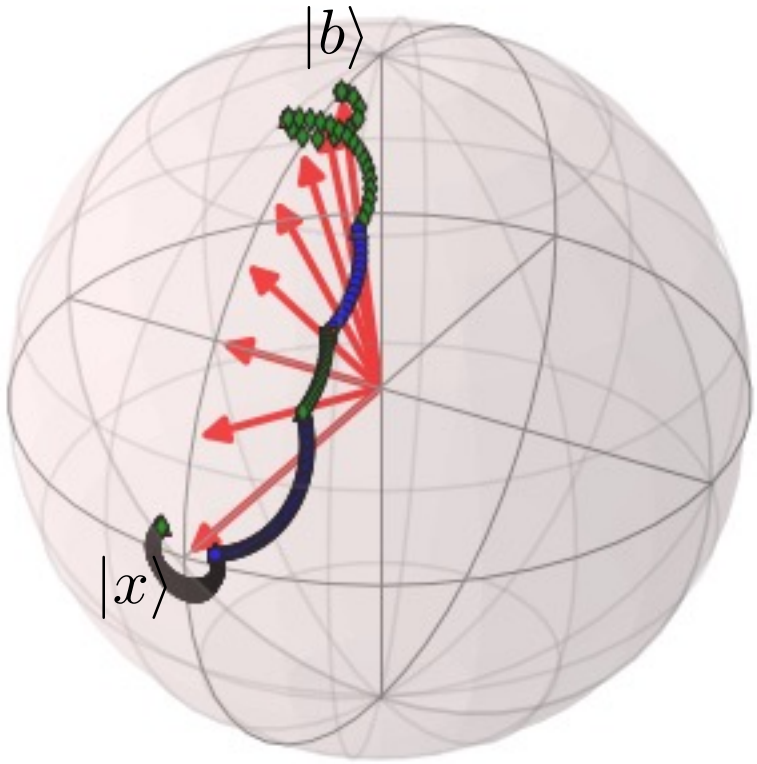
Example

Sample run of the Randomization Method



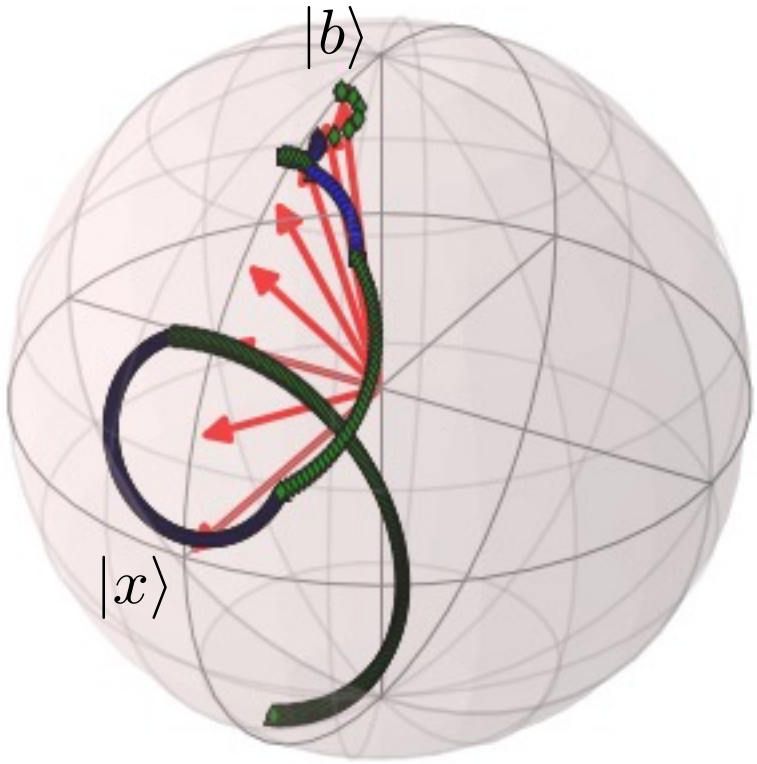
Example

Sample run of the Randomization Method



Example

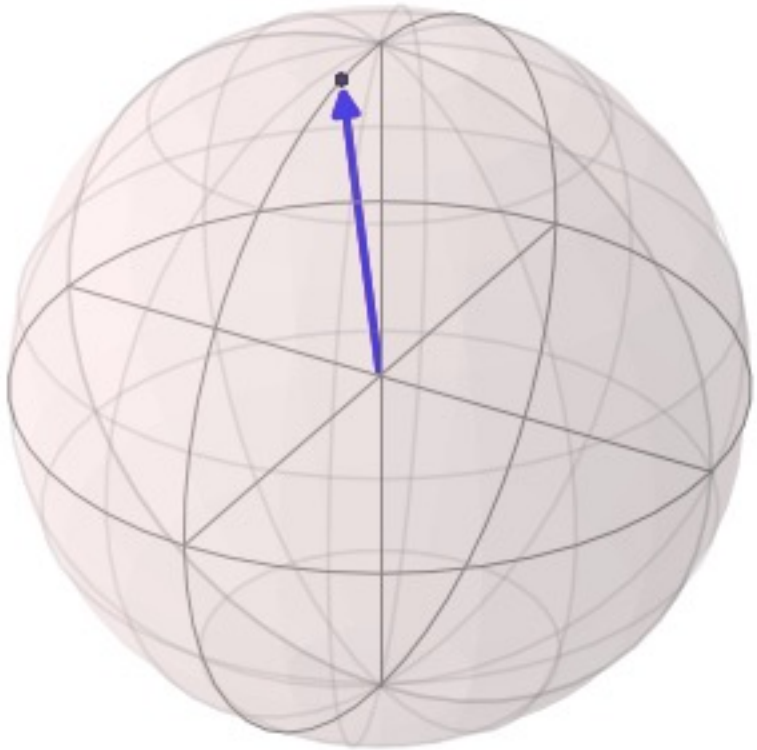
Sample run of the Randomization Method



Example

Average of 100 runs

$s_0 = 0$



exact ground state

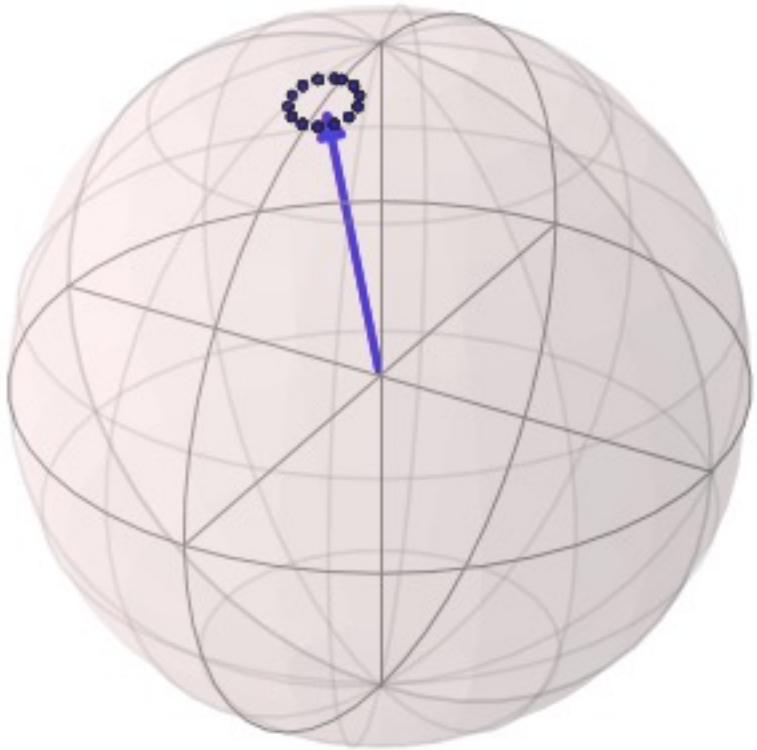
mixed state generated by the algorithm

pure state generated at each run of RM

Example

Average of 100 runs

$s_1 \approx 0.36$



exact ground state

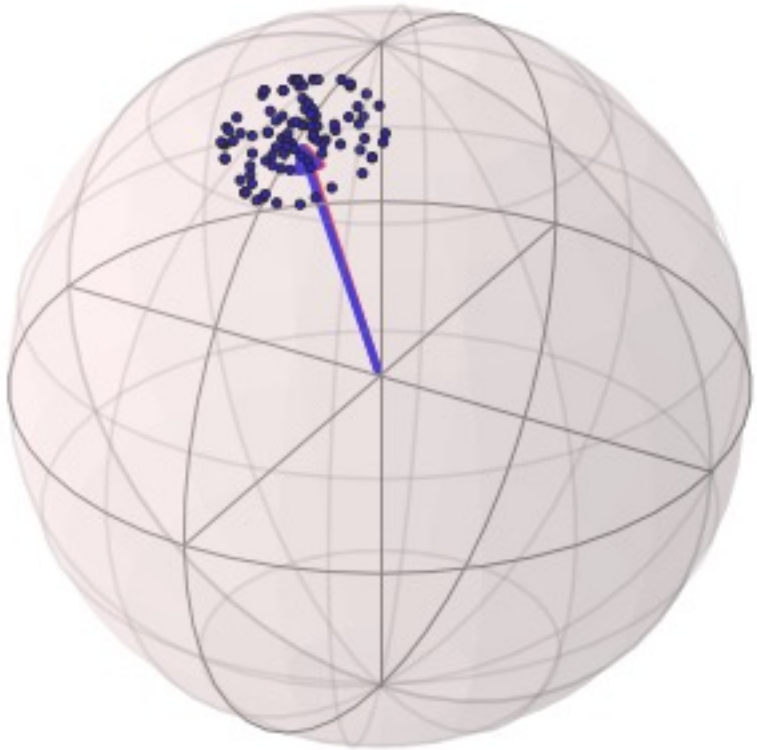
mixed state generated by the algorithm

pure state generated at each run of RM

Example

Average of 100 runs

$s_2 \approx 0.59$



exact ground state

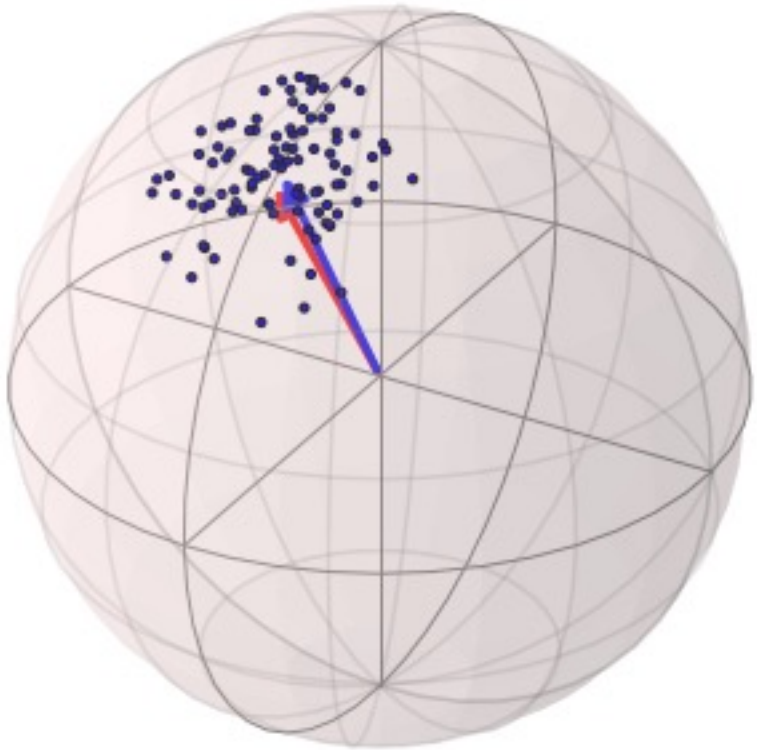
mixed state generated by the algorithm

pure state generated at each run of RM

Example

Average of 100 runs

$s_3 \approx 0.74$



exact ground state

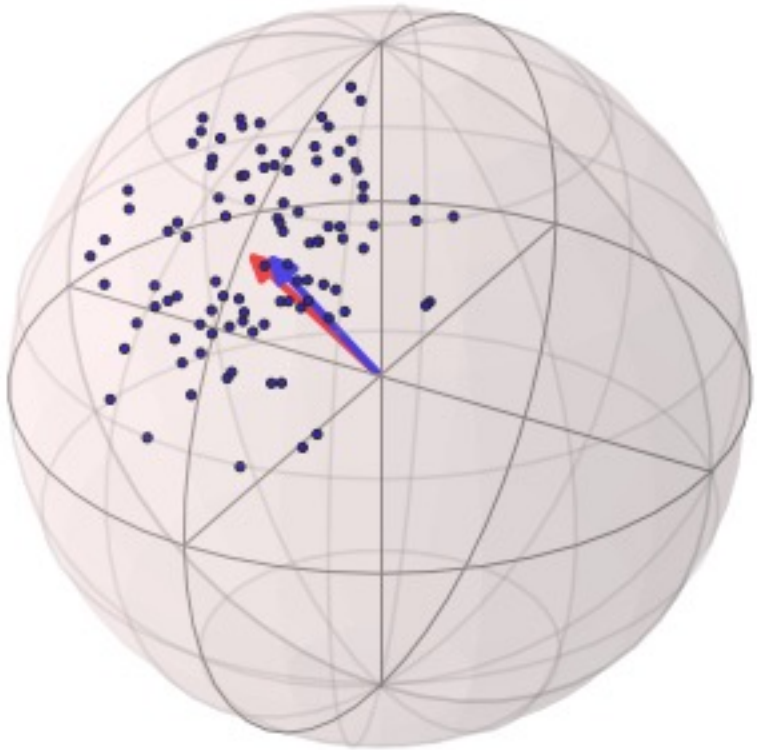
mixed state generated by the algorithm

pure state generated at each run of RM

Example

Average of 100 runs

$s_4 \approx 0.83$



exact ground state

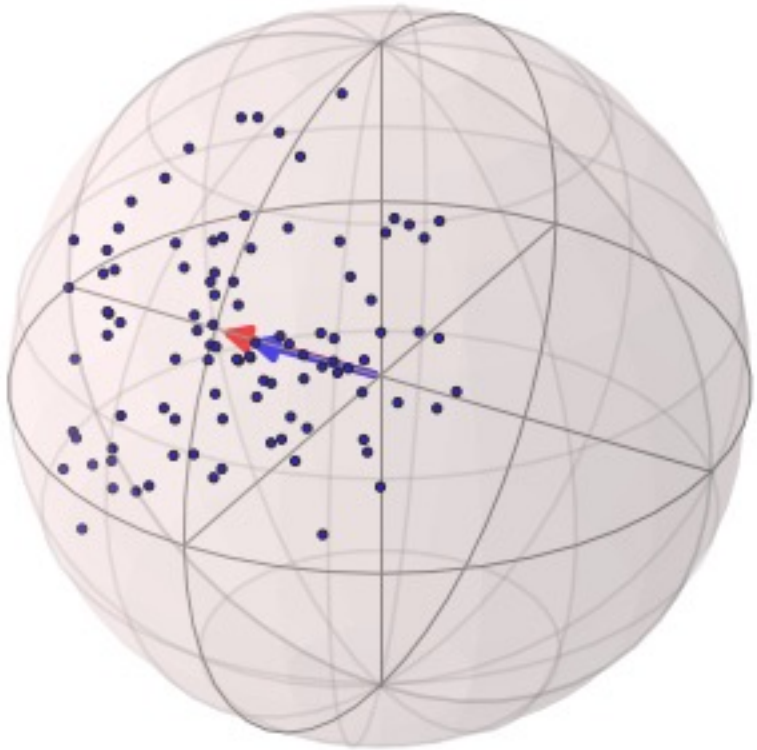
mixed state generated by the algorithm

pure state generated at each run of RM

Example

Average of 100 runs

$s_5 \approx 0.90$



exact ground state

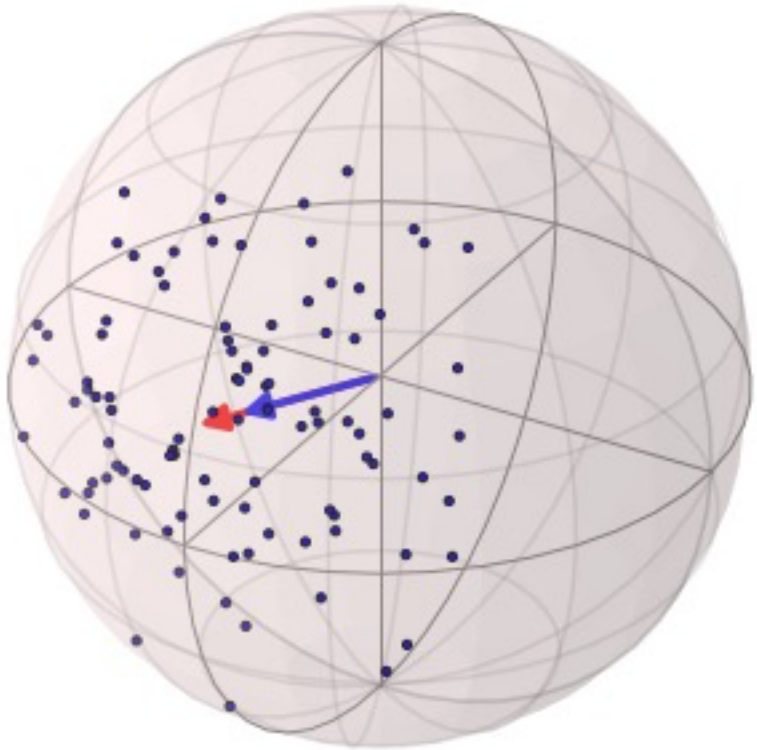
mixed state generated by the algorithm

pure state generated at each run of RM

Example

Average of 100 runs

$s_6 \approx 0.96$



exact ground state

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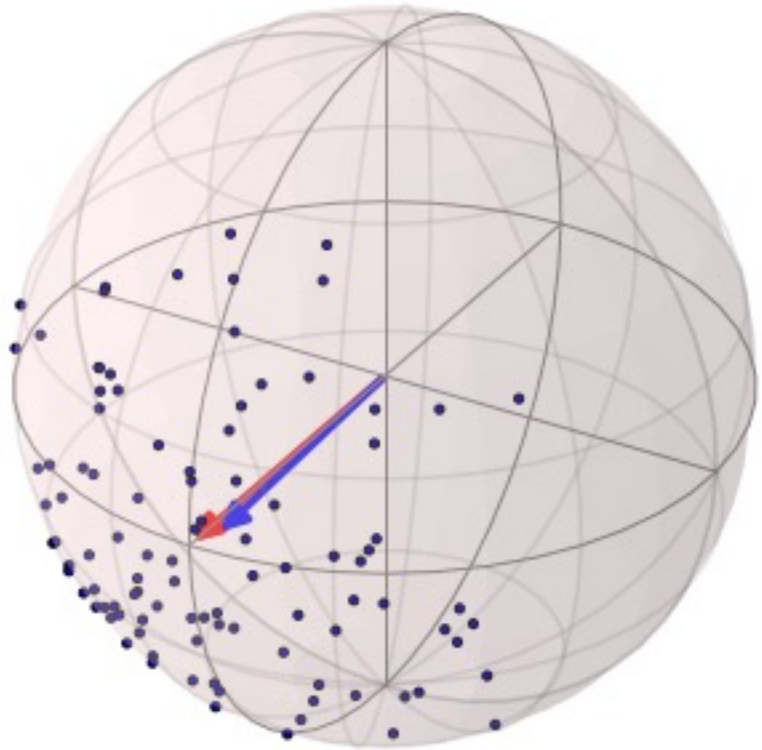
Example

Average of 100 runs

$s_7 \approx 1.0$



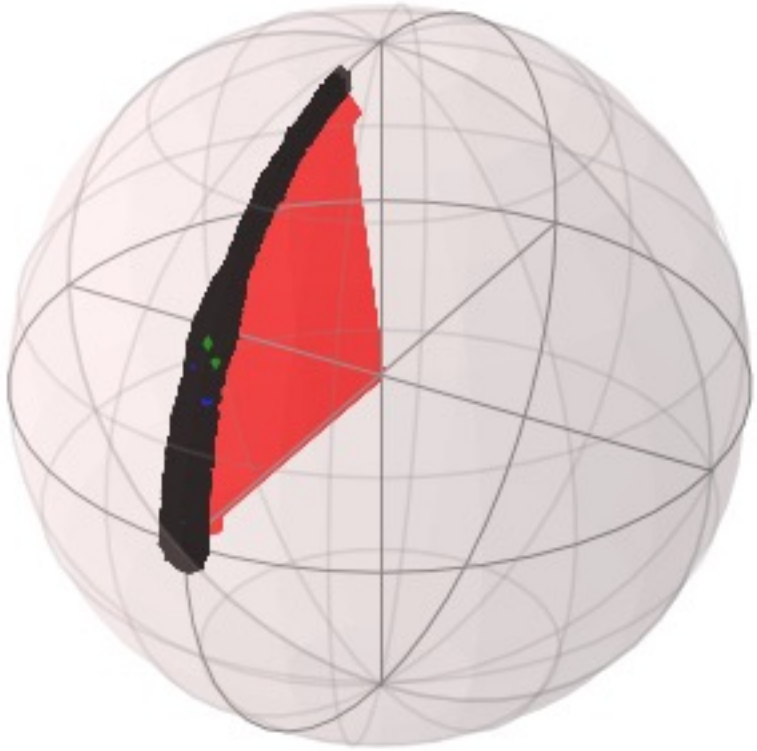
exact ground state

mixed state generated by the algorithm

pure state generated at each run of RM

Example

$$q = 1000$$



Conclusions

- Our algorithms are conceptually simple and don't rely on complex subroutines. They only use Hamiltonian simulation.
- The Hamiltonians involved in our algorithms are easily described in terms of the inputs of the problem and can be efficiently simulated in a digital QC.
- At most 2 additional ancillas are needed, beyond what is necessary for Hamiltonian simulation.
- Phase estimation and VTAA, on the other hand, require several ancillary qubits (beyond what is necessary for Hamiltonian simulation) and a lot of controlled operations.
- Our results emphasize the importance of considering models of quantum computing, which go beyond the gate-based model, for discovering novel quantum algorithms.

A variational algorithm

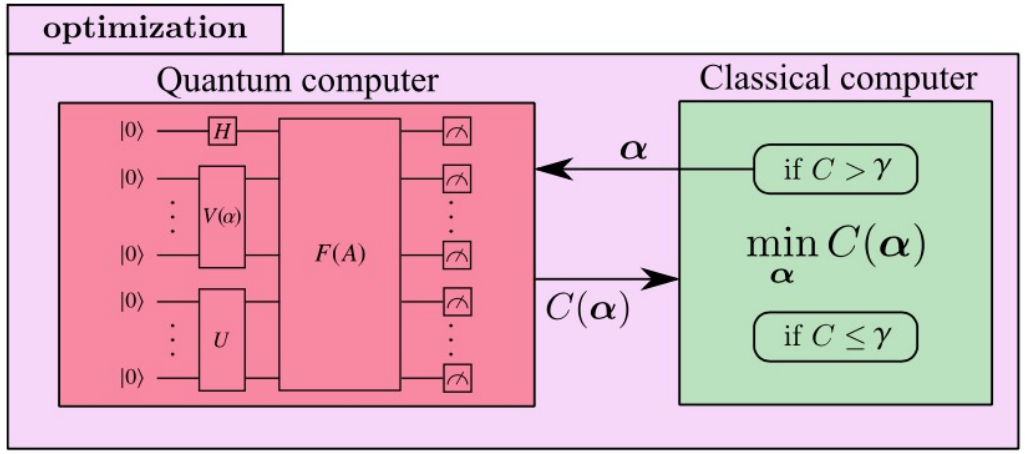
Variational Quantum Linear Solver

Carlos Bravo-Prieto,[1,2,3] Ryan LaRose,[4] M. Cerezo,[1,5] Yiğit Subaşı,[6] Lukasz Cincio,[1] and Patrick J. Coles[1,*]

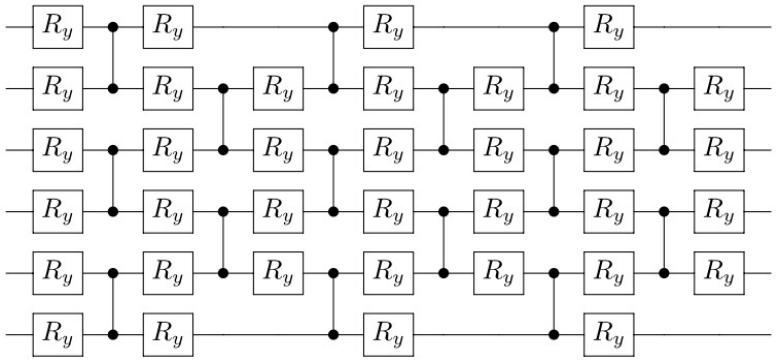
[1] *Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA.*
[2] *Barcelona Supercomputing Center, Barcelona, Spain.*
[3] *Institut de Ciències del Cosmos, Universitat de Barcelona, Barcelona, Spain.*
[4] *Department of Computational Mathematics, Science, and Engineering & Department of Physics and Astronomy, Michigan State University, East Lansing, MI 48823, USA.*
[5] *Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM, USA*
[6] *Computer, Computational and Statistical Sciences Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*

Previously proposed quantum algorithms for solving linear systems of equations cannot be implemented in the near term due to the required circuit depth. Here, we propose a hybrid quantum-classical algorithm, called Variational Quantum Linear Solver (VQLS), for solving linear systems on near-term quantum computers. VQLS seeks to variationally prepare $|x\rangle$ such that $A|x\rangle \propto |b\rangle$. We derive an operationally meaningful termination condition for VQLS that allows one to guarantee that a desired solution precision ϵ is achieved. Specifically, we prove that $C \geqslant \epsilon^2/\kappa^2$, where C is the VQLS cost function and κ is the condition number of A. We present efficient quantum circuits to estimate C, while providing evidence for the classical hardness of its estimation. Using Rigetti's quantum computer, we successfully implement VQLS up to a problem size of 1024×1024. Finally, we numerically solve non-trivial problems of size up to $2^{50} \times 2^{50}$. For the specific examples that we consider, we heuristically find that the time complexity of VQLS scales efficiently in ϵ, κ, and the system size N.

arXiv:1909.05820v2



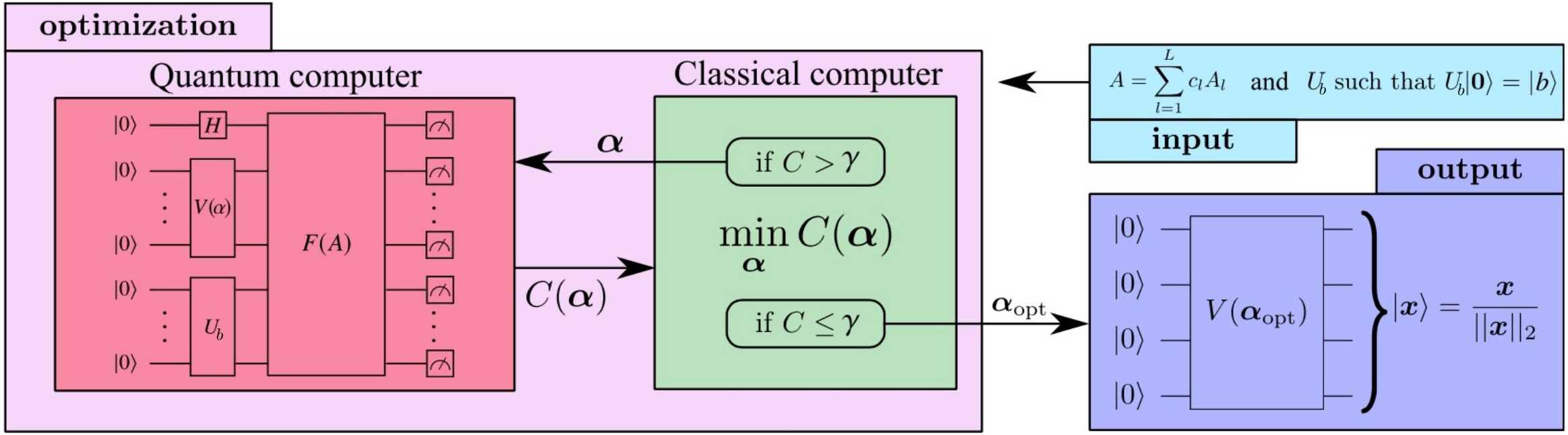
$$V(\boldsymbol{\alpha}) =$$



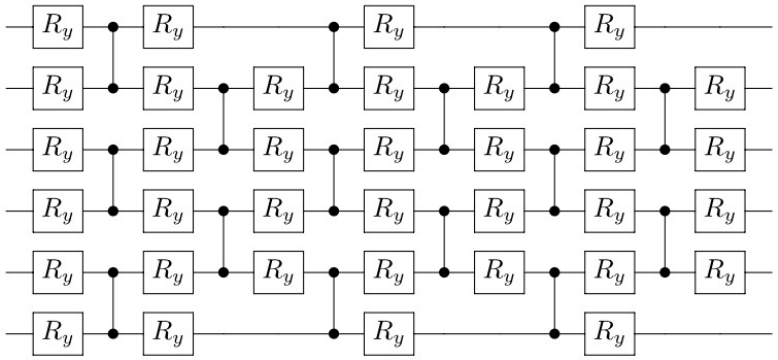




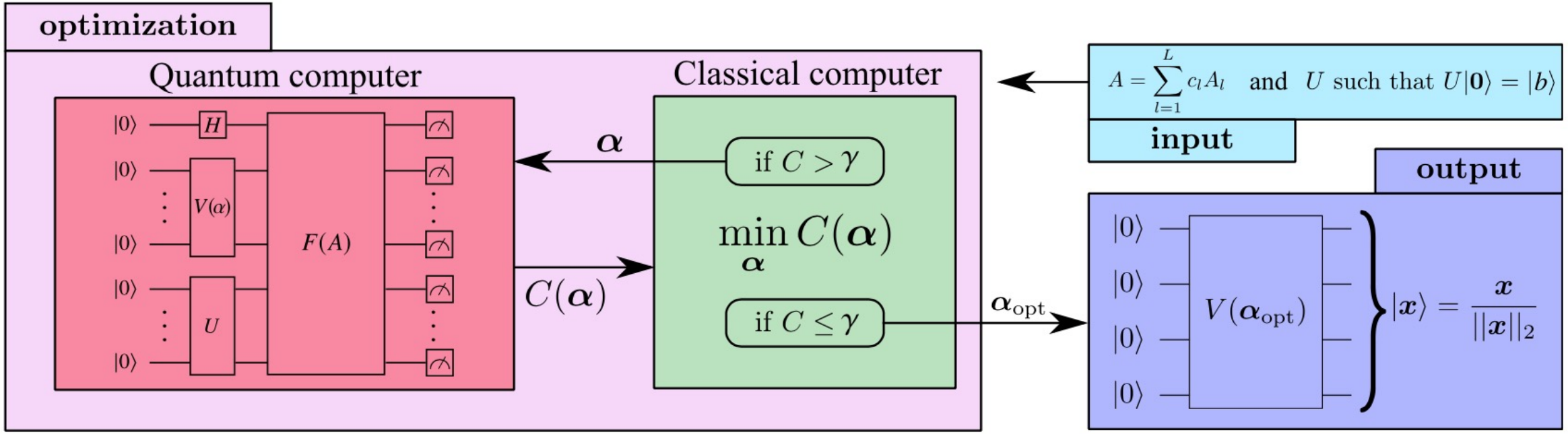
Variational quantum linear solver



$V(\boldsymbol{\alpha}) =$



Variational quantum linear solver



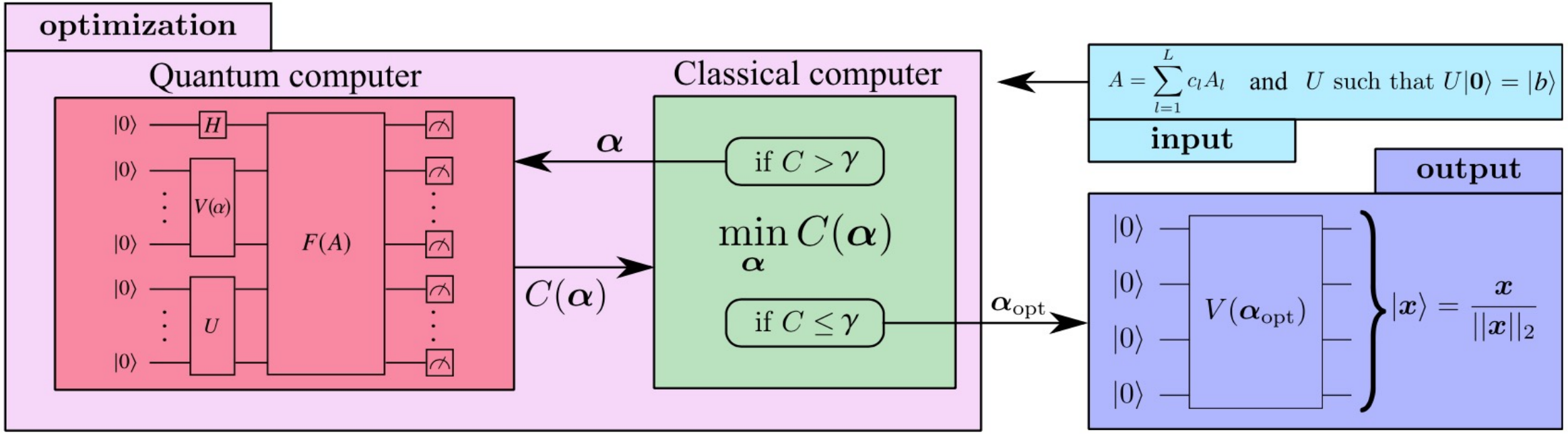
The expectation value of the final Hamiltonian of the adiabatic approach is a valid cost function.

$$H = AP_b^{\perp}A$$

$$|x(\boldsymbol{\alpha})\rangle = V(\boldsymbol{\alpha})|0\rangle \qquad C_G(\boldsymbol{\alpha}) = \langle x(\boldsymbol{\alpha})\,|H|\,x(\boldsymbol{\alpha})\rangle$$

$$C_G = 0 \Longleftrightarrow A|x(\boldsymbol{\alpha})\rangle \propto |b\rangle$$

Variational quantum linear solver



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$$H = AP_b^{\perp}A$$

$$C_G(\boldsymbol{\alpha}) = \langle x(\boldsymbol{\alpha})\,|H|\,x(\boldsymbol{\alpha})\rangle$$

$$C_G = 0 \Longleftrightarrow A|x(\boldsymbol{\alpha})\rangle \propto |b\rangle$$

Cost functions with better trainability have also been constructed by requiring local closeness of the quantum states $A\,|x\rangle$ *and* $|b\rangle$.

Operational Meaning of Cost Function

It can be shown that

$$\kappa^2 C_G \geq D(|x\rangle\langle x|, |x(\boldsymbol{\alpha})\rangle\langle x(\boldsymbol{\alpha})|)^2$$

This means that in order to guarantee an error less than ϵ, the expectation value of the Hamiltonian has to be confirmed to be smaller than

$$\frac{\epsilon^2}{\kappa^2} \geq C_G = \langle x(\boldsymbol{\alpha})|H|x(\boldsymbol{\alpha})\rangle$$

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$$\kappa^2 C_G \geq D(|x\rangle\langle x|, |x(\boldsymbol{\alpha})\rangle\langle x(\boldsymbol{\alpha})|)^2$$

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$$\frac{\epsilon^2}{\kappa^2} \geq C_G = \langle x(\boldsymbol{\alpha}) | H | x(\boldsymbol{\alpha}) \rangle$$

Because of sampling noise, this means the number of runs scales as:

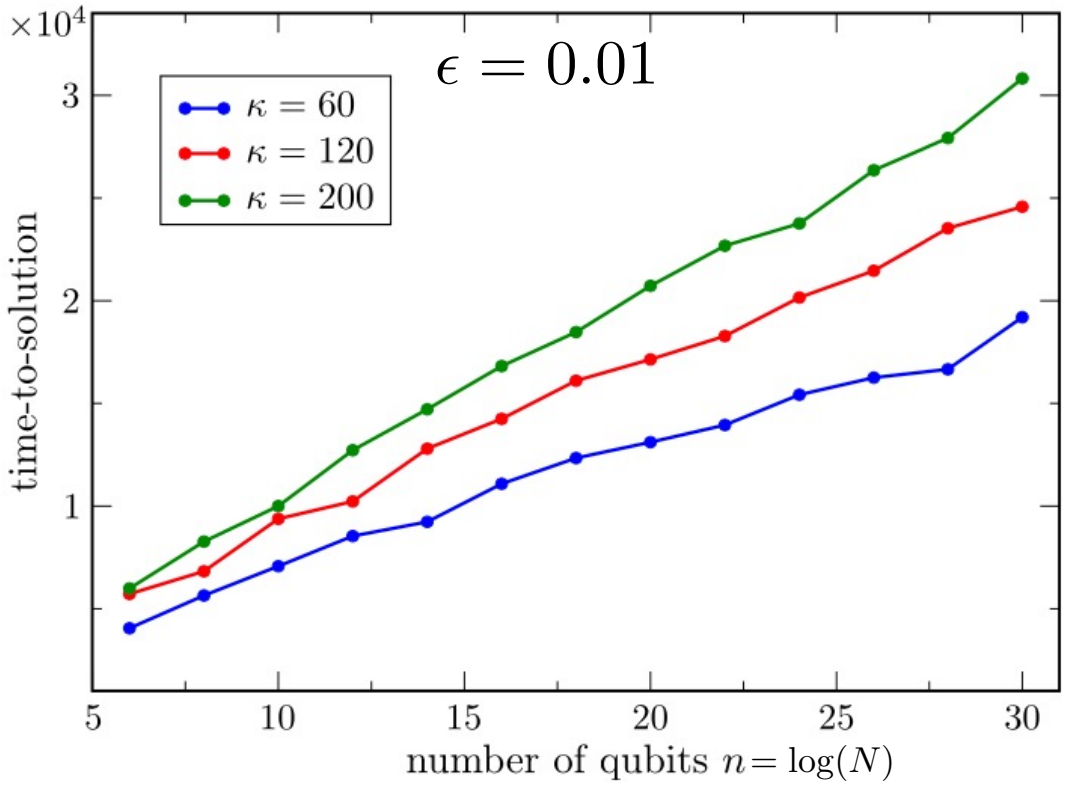
$$N_{\text{runs}} \sim \frac{\kappa^4}{\epsilon^4}$$

Variational quantum linear solver
Numerical Experiments

$$A = \frac{1}{\zeta}\left(\sum_{j=1}^{n}\sigma_j^X + J\sum_{j=1}^{n-1}\sigma_j^Z\sigma_{j+1}^Z + \eta\mathbf{1}\right)$$

$$|b\rangle = H^{\otimes n}|0\rangle$$

Number of iterations needed to guarantee desired precision is linear in n, i.e. log(N).

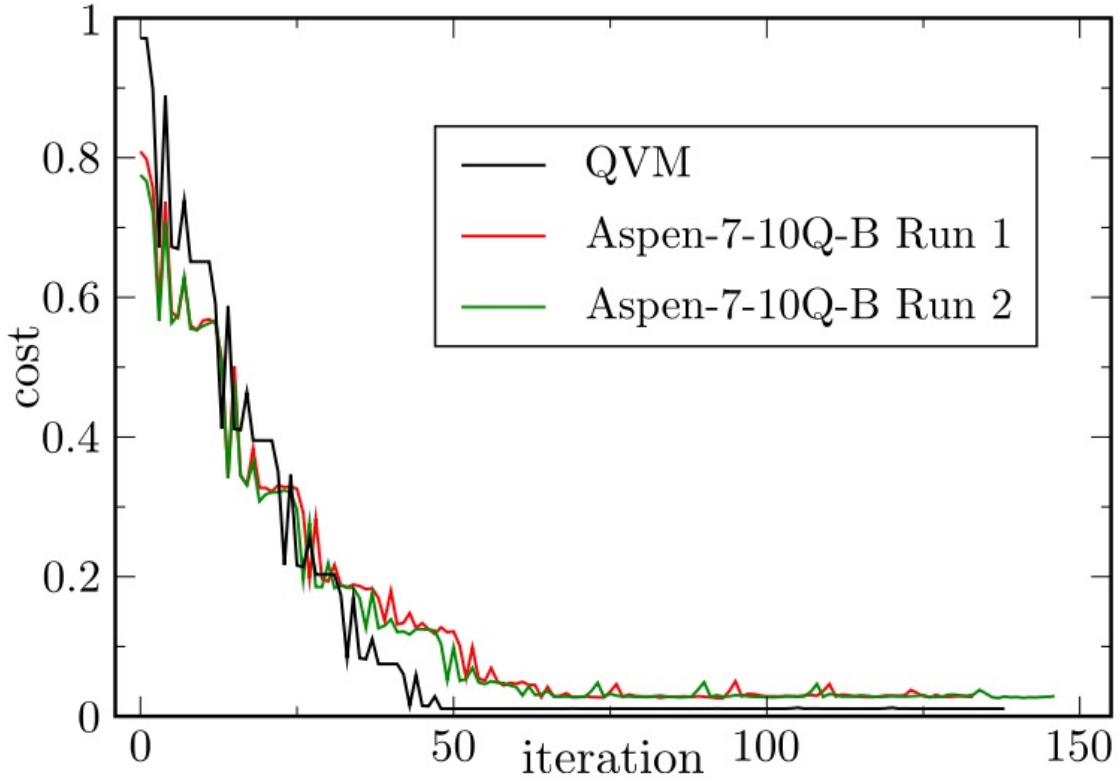


Variational quantum linear solver
Experiments on Rigetti's QPU

$$A = \frac{1}{\zeta}\left(\sum_{j=1}^{n}\sigma_j^X + J\sum_{j=1}^{n-1}\sigma_j^Z\sigma_{j+1}^Z + \eta\mathbf{1}\right)$$

(1024x1024)

$$|b\rangle = H^{\otimes n}|0\rangle$$



Conclusions

- Largest implementation on real hardware: $n = 10$ qubits, 1024×1024
- Numerical solution to non-trivial problems of size up to $2^{50} \times 2^{50}$
- Efficient circuits for cost function evaluation
- Trainable local cost functions
- Conditions that guarantee desired error
- Heuristically analyzed scaling with respect to $\epsilon,\ \kappa,\ N$.

Complexity of verification

PRX QUANTUM **2,** 010315 (2021)

Complexity of Quantum State Verification in the Quantum Linear Systems Problem

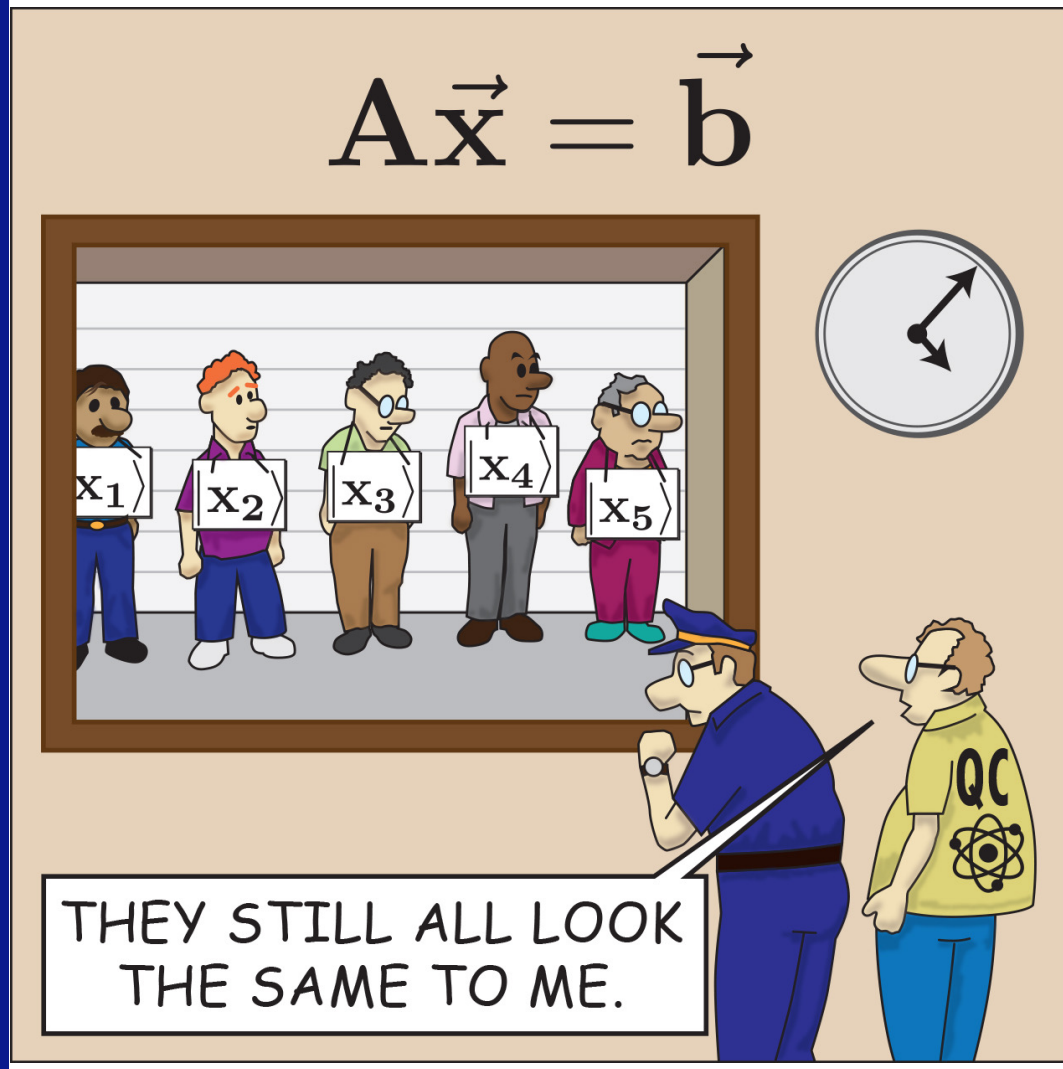
Rolando D. Somma[1,*] and Yiğit Subaşı[2,†]

[1] *Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

[2] *Computer, Computational, and Statistical Sciences Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

(Received 14 August 2020; accepted 16 December 2020; published 27 January 2021)

We analyze the complexity of quantum state verification in the context of solving systems of linear equations of the form $A\vec{x} = \vec{b}$. We show that any quantum operation that verifies whether a given quantum state is within a constant distance from the solution of the quantum linear systems problem requires $q = \Omega(\kappa)$ uses of a unitary that prepares a quantum state $|b\rangle$, proportional to $\vec{b}$, and its inverse in the worst case. Here, κ is the condition number of matrix A. For typical instances, we show that $q = \Omega(\sqrt{\kappa})$ with high probability. These lower bounds are almost achieved if quantum state verification is performed using known quantum algorithms for the quantum linear systems problem. We also analyze the number of copies of $|b\rangle$ required by verification procedures of the prepare-and-measure type. In this case, the lower bounds are quadratically worse, being $\Omega(\kappa^2)$ in the worst case and $\Omega(\kappa)$ in typical instances with high probability. We discuss the implications of our results to known variational and related approaches to this problem, where state preparation, gate, and measurement errors will need to decrease rapidly with κ for worst-case and typical instances if error correction is not used, and present some open problems.

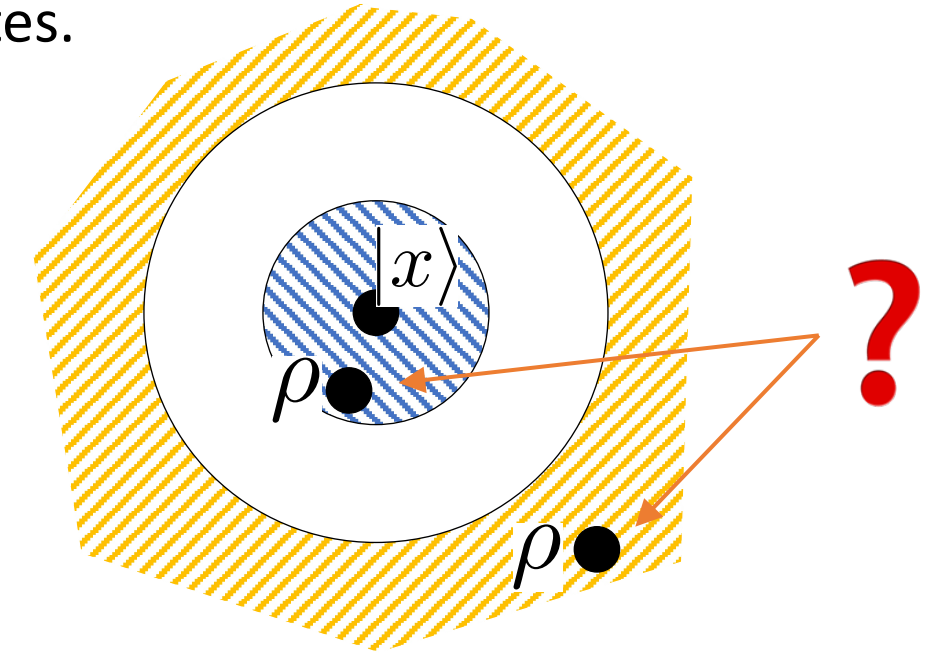


Quantum State Verification (QSV) Problem

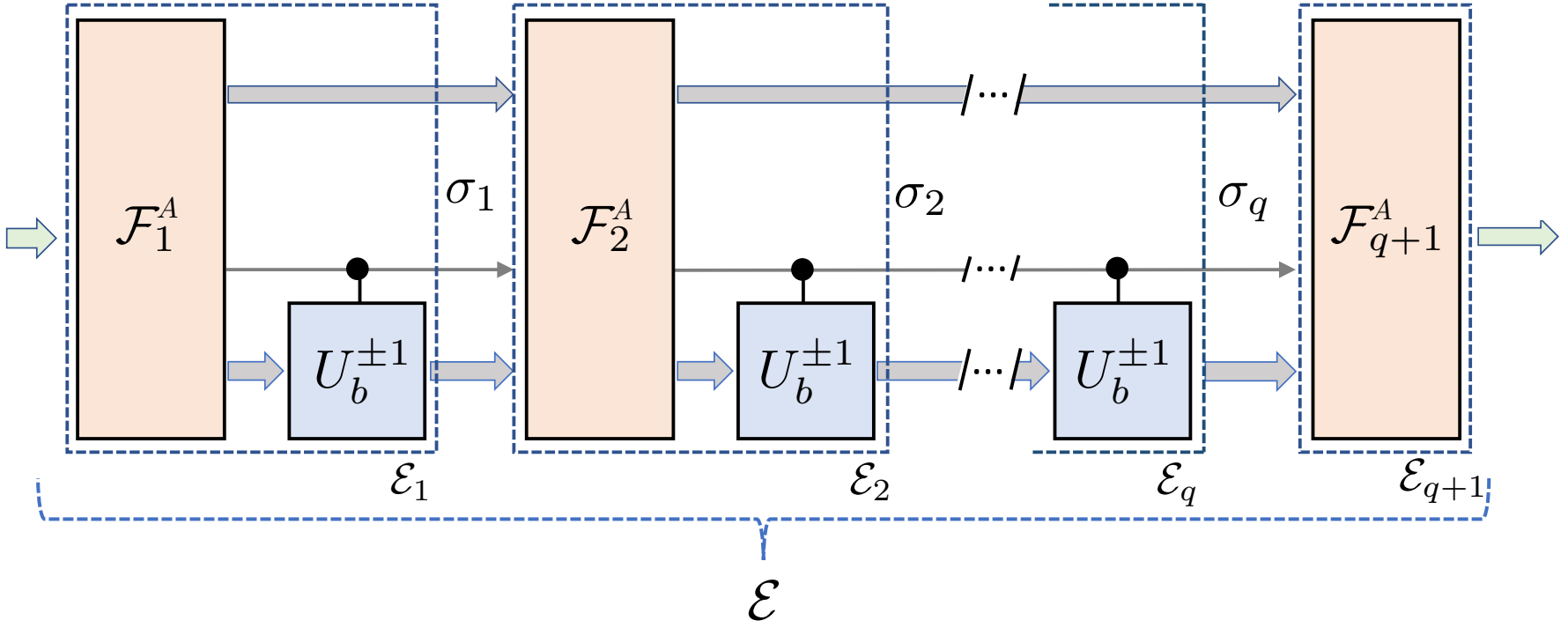
Given arbitrarily many copies of ρ, output a random bit r as follows:

$$\Pr(r=1) \begin{cases} \geq 2/3 & \text{if} \quad D_{\rho,x} \leq 1/8 \,, \\ \leq 1/3 & \text{if} \quad D_{\rho,x} > 1/2 \,. \end{cases}$$

where $D_{\rho,x} = \frac{1}{2}\mathrm{Tr}|\rho - |x\rangle\langle x||$ is the trace distance which is a measure of distinguishability between two quantum states.



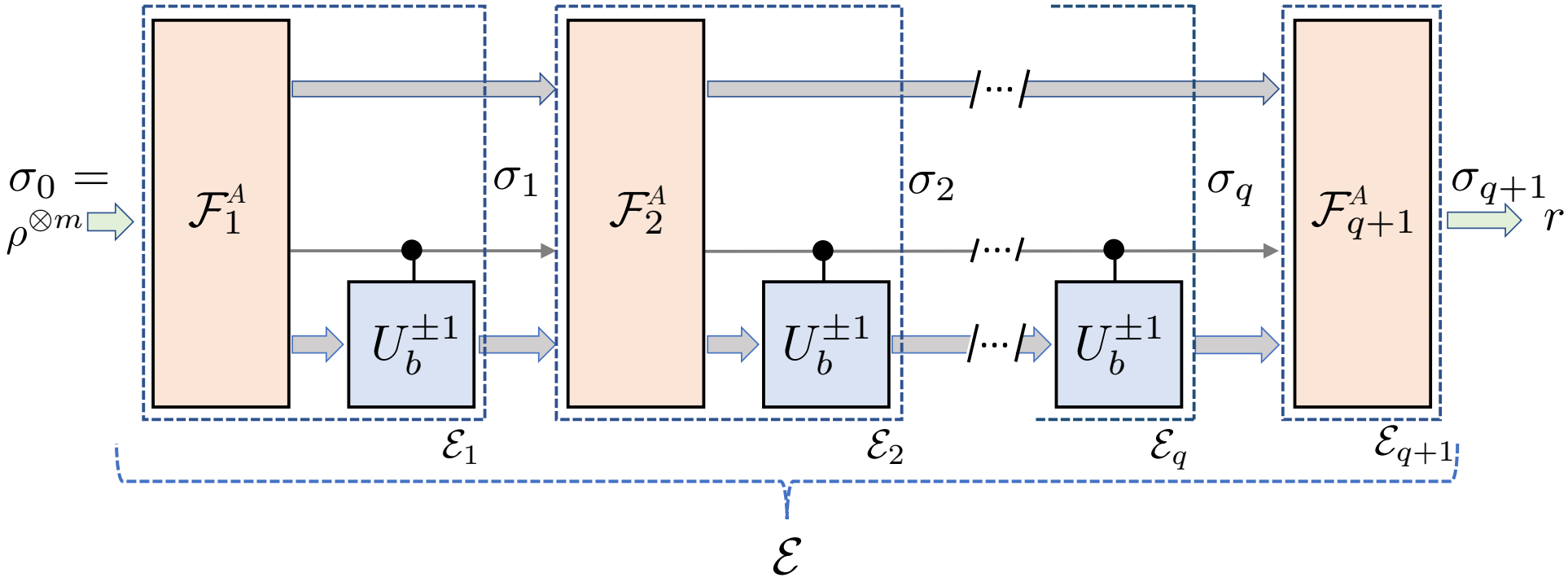
Protocol for QSV



Inputs: A and $\vec{b}$

Output: A quantum operation $\mathcal{E}$

Protocol for QSV

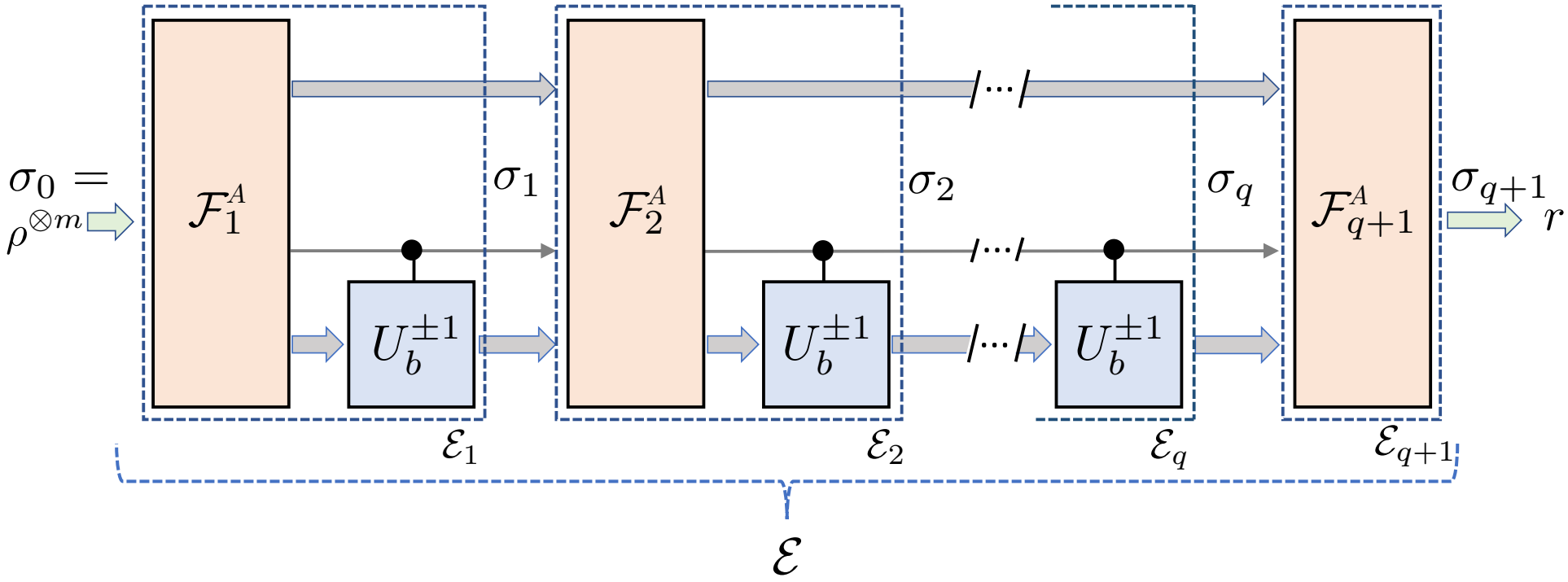


Inputs: A and $\vec{b}$

Output: A quantum operation $\mathcal{E}$

$\mathcal{E}$ takes as input m copies of ρ and outputs a single bit r

Protocol for QSV



We want:

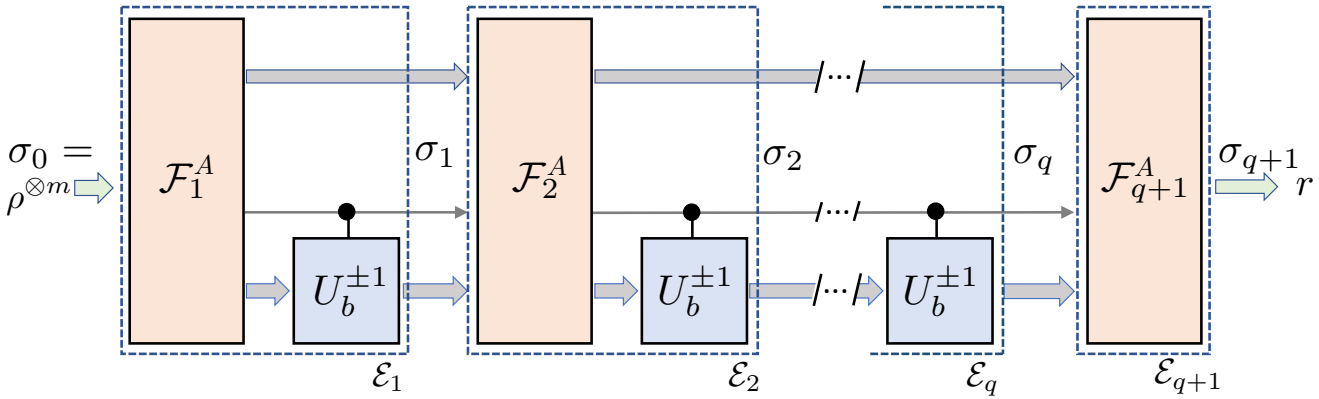
$$\Pr(r=1) \begin{cases} \geq 2/3 & \text{if} \quad D_{\rho,x} \leq 1/8 \ , \\ \leq 1/3 & \text{if} \quad D_{\rho,x} > 1/2 \ . \end{cases}$$

Arbitrary Instances

Main result: Consider any instance of the QLSP, specified by A and $\vec{b}$, and any protocol for QSV as described. Then, for all quantum states ρ that satisfy $D_{\rho,x} \leq 1/8$, the number of $cU_b^{\pm 1}$'s required to implement $\mathcal{E}$ on input $\sigma_0 = \rho^{\otimes m}$ satisfies

$$\Pr\left(q_{A,b} > \frac{1}{13}\,\frac{\kappa}{\|A^{-1}\,|b\rangle\,\|}\right) \geq \frac{1}{6}\,.$$

Number of $cU_b^{\pm 1}$ needed for QSV for a given instance.



Worst-Case

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$$1 \leq \|A^{-1}|b\rangle\| \leq \kappa$$

worst case

best case

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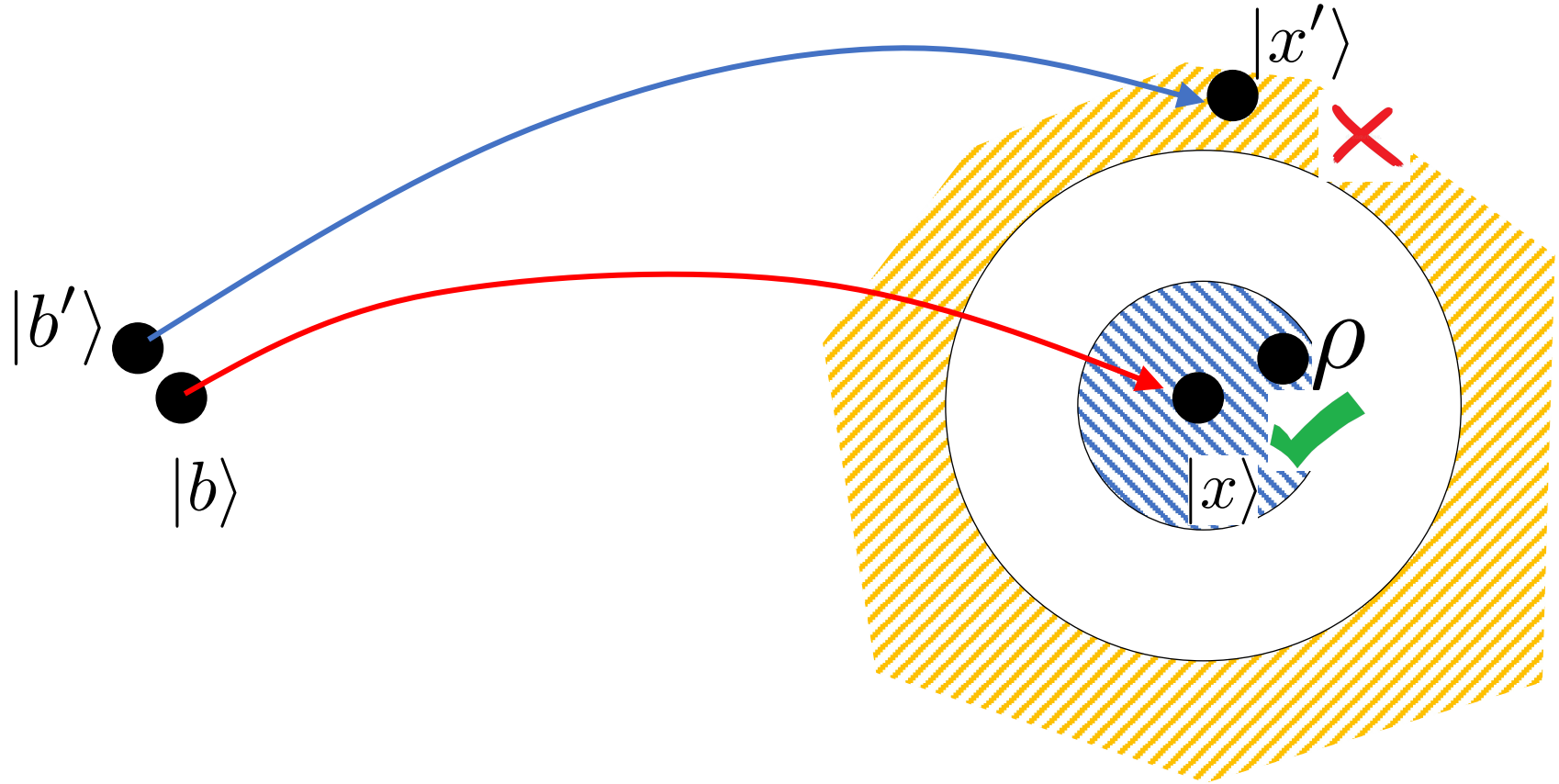
Verification of solution is asymptotically as hard as solving the problem!

Outline of Proof (Worst-Case)

For given input state $|b\rangle$ we construct another state $|b'\rangle$ which is close to it, but such that the corresponding solutions of QLSP, $|x\rangle$ and $|x'\rangle$ are as far from each other as possible.



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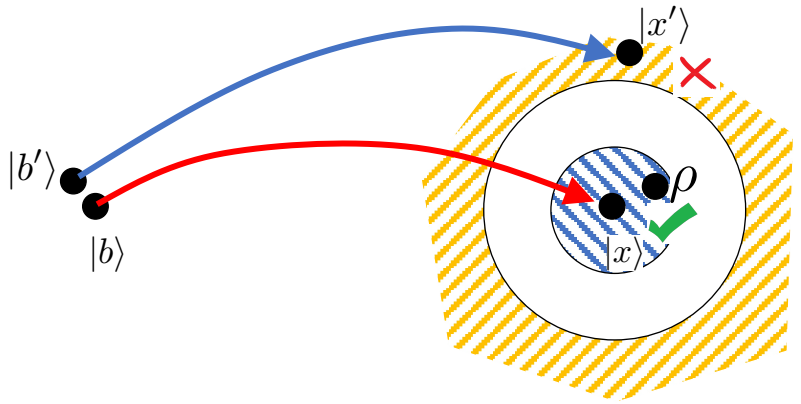
$$|b\rangle \propto |\lambda = 1\rangle + \frac{1}{\kappa}|\lambda = \frac{1}{\kappa}\rangle \quad \longrightarrow \quad |x\rangle \propto |\lambda = 1\rangle + |\lambda = \frac{1}{\kappa}\rangle$$

A similar instance is given by:

$$|b'\rangle \propto |\lambda = 1\rangle - \frac{1}{\kappa}|\lambda = \frac{1}{\kappa}\rangle \quad \longrightarrow \quad |x'\rangle \propto |\lambda = 1\rangle - |\lambda = \frac{1}{\kappa}\rangle$$



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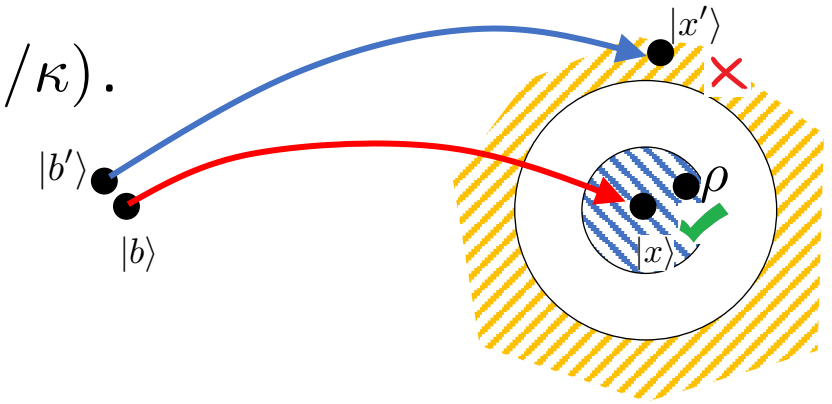
Also $\| |b'\rangle - |b\rangle \| = O(1/\kappa) \implies \exists \; \|U_{b'} - U_b\| = O(1/\kappa)$.

If $\rho = |x\rangle\langle x|$: QSV for $A\vec{x} = \vec{b}$ should accept ρ

QSV for $A\vec{x'} = \vec{b'}$ should reject ρ



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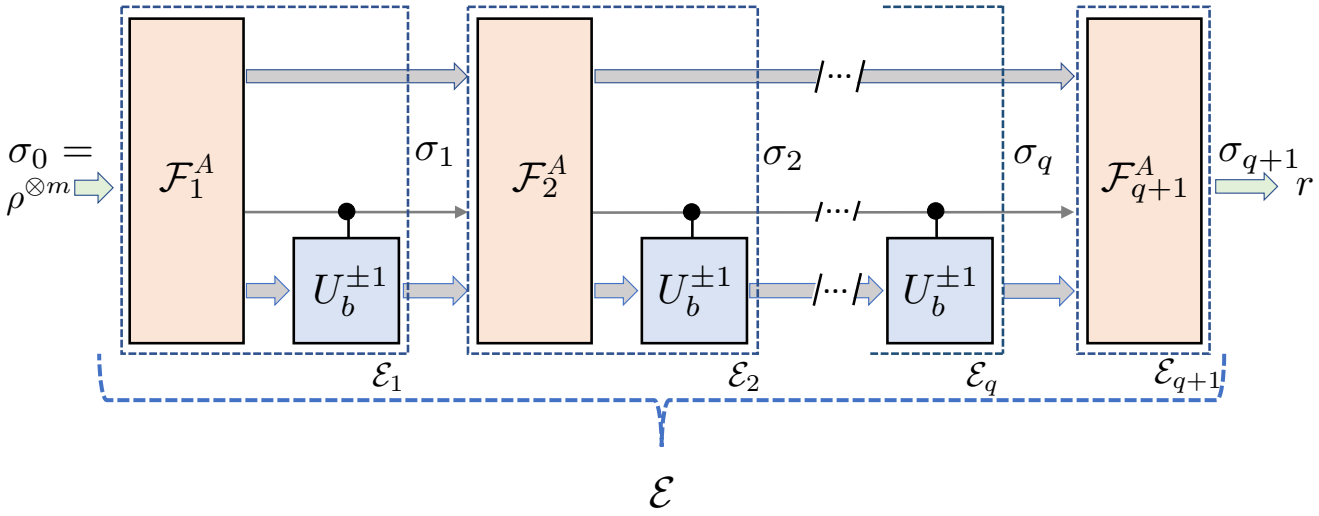
QSV for A and b



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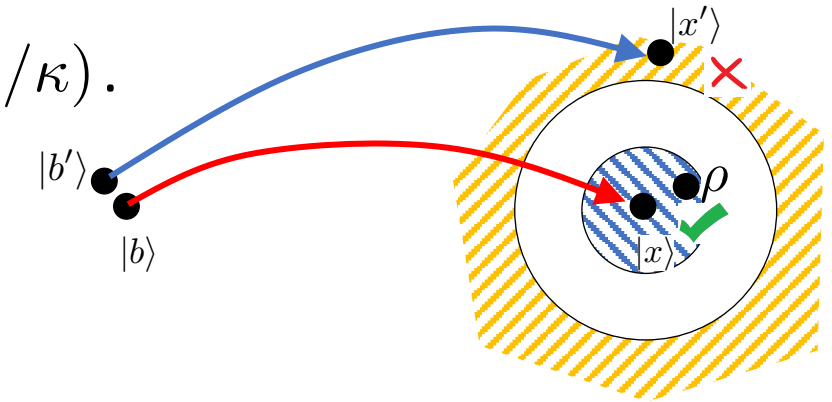

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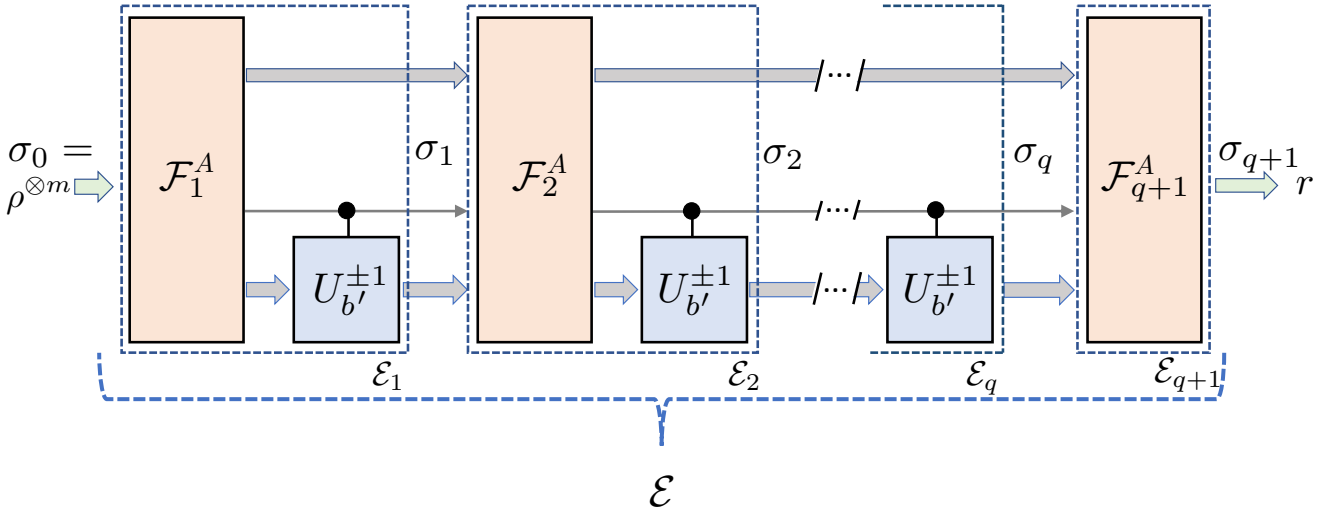

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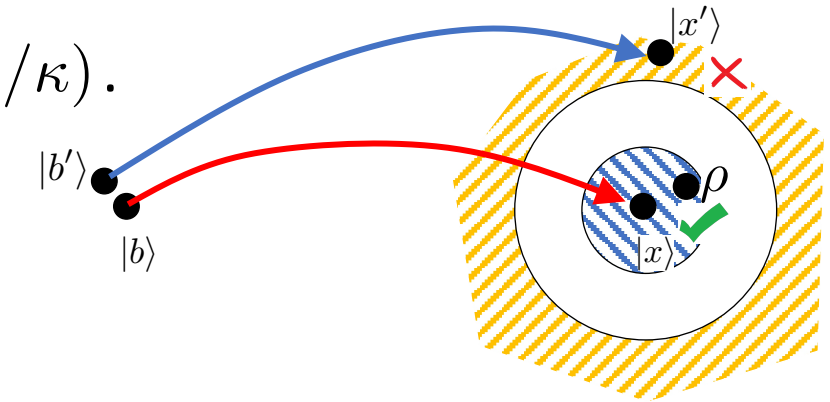

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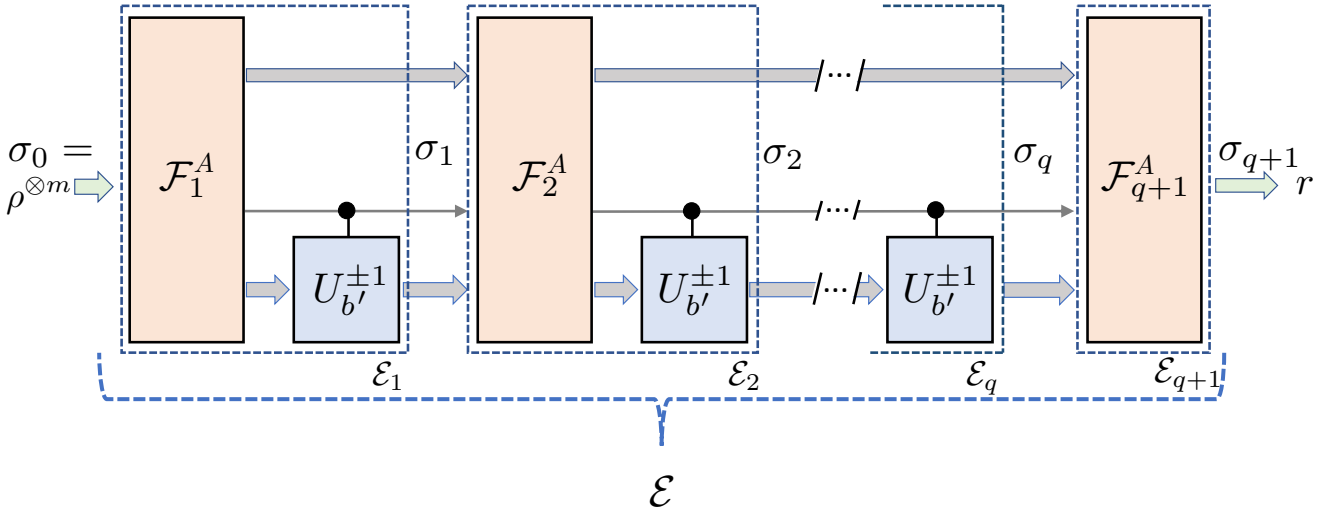

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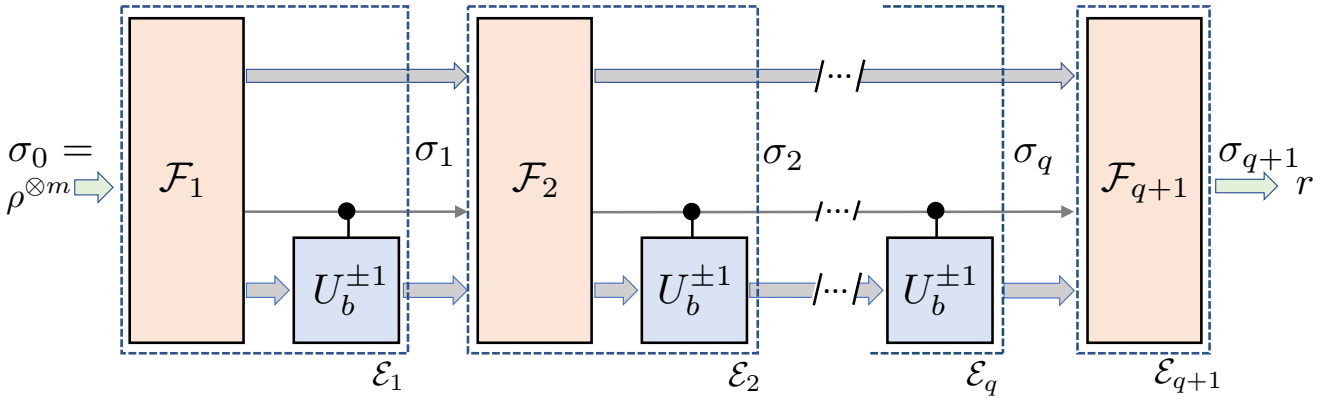
QSV alg. has to query the oracle $\Omega(\kappa)$ times.

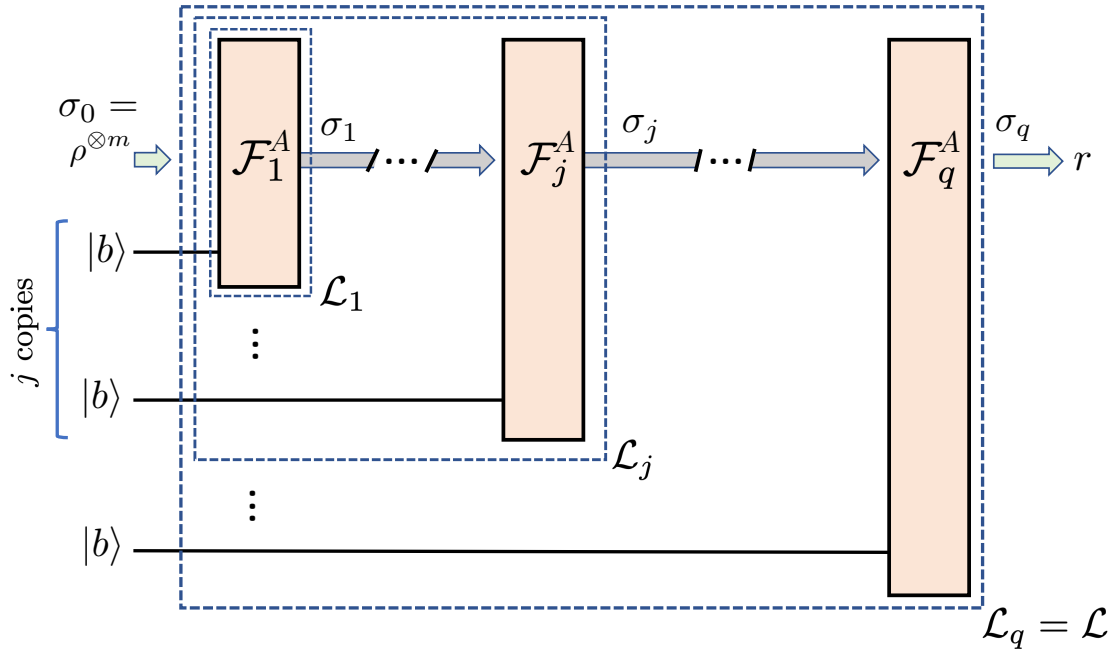
Typical Instances

For typical instances we show that $\|A^{-1}|b\rangle\| \sim \sqrt{\kappa}$

$$\Pr\left(q_{A,b} > \frac{1}{16}\sqrt{\kappa}\right) \geq \frac{1 - 4e^{-cN/\kappa}}{6} \ .$$



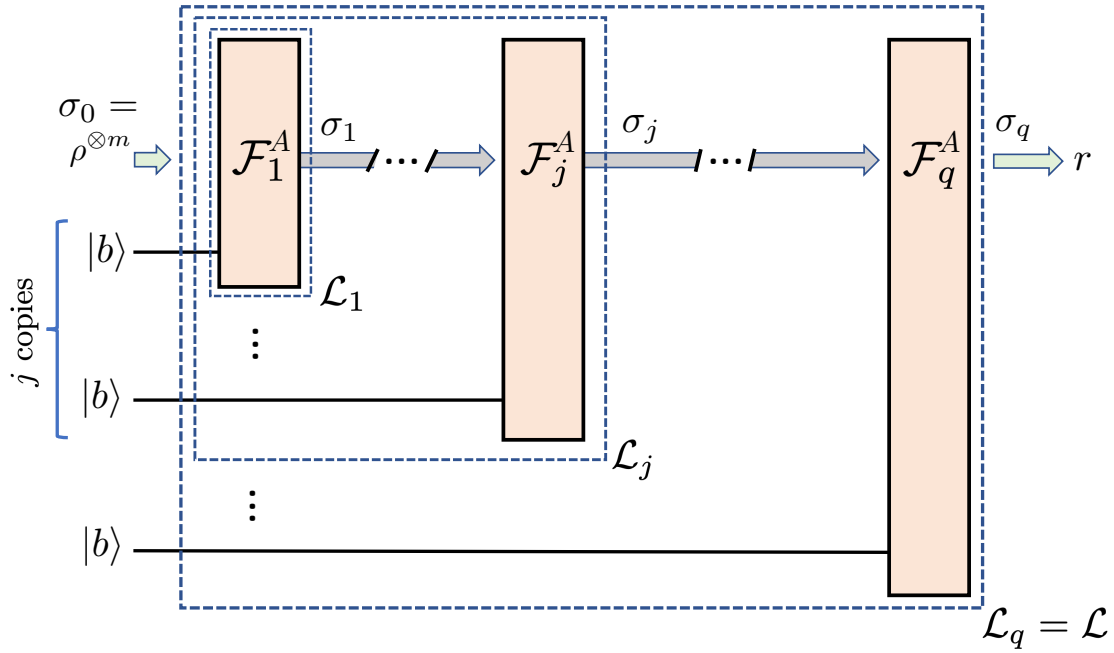
Prepare and Measure Scheme



Prepare and Measure: $\Pr\left(q_{A,b} > \frac{1}{150}\frac{\kappa^2}{\|A^{-1}|b\rangle\|^2}\right) \geq \frac{1}{6}$.

Number of copies of $|b\rangle$ needed for QSV for a given instance.

Prepare and Measure Scheme



Prepare and Measure: $\Pr\left(q_{A,b} > \frac{1}{150}\frac{\kappa^2}{\|A^{-1}|b\rangle\|^2}\right) \geq \frac{1}{6}$.

Recall: Using Unitaries: $\Pr\left(q_{A,b} > \frac{1}{13}\frac{\kappa}{\|A^{-1}|b\rangle\|}\right) \geq \frac{1}{6}$. (Quadratically better)

Conclusions

- Our results place limitations for QLSP algorithms that require a verification step, such as known variational algorithms.
- Even disregarding the complexity of the optimization loop, the asymptotic complexity of the verification step alone is at least that of optimal algorithms for solving QLSP.
- For example, high precision cost function evaluation necessitates quantum circuits to be executed many times and with high fidelity. The latter suggests that quantum error correction will be needed.
- $cU_b^{\pm 1}$ is treated as a "black box" and no assumption is made on the inner workings of such unitaries. If such knowledge is provided, it may be exploited for more efficient QSV and for solving the QLSP faster.

Future directions:

- Relaxations of the QLSP that circumvent our bounds (i.e. goal is not to prepare $|x\rangle$).
- Analyze the complexity of QSV in terms of resources other than U_b.
- Generalize and apply our proof technique to related state-verification problems.

Thank You!